A RANDOM DEGREE WITH STRONG MINIMAL COVER

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ABSTRACT. We show that there exists a Martin-Löf random degree which has a strong minimal cover.

1. Introduction

The Turing degree of an infinite binary sequence may be regarded as one way of measuring its complexity. If an infinite sequence A is of degree below that of B (denoted $A \leq_T B$) then B is at least as difficult to compute as A is, and so, is this sense, is at least as complicated. It is a basic and natural question, then, to ask how the Turing degree of a sequence relates to other measures of complexity. The study of algorithmic randomness has recently been a very active area of research and so a fundamental issue with which we are concerned is as to what can be said about the relationship between Turing complexity and randomness.

One of the most attractive aspects of the study of algorithmic randomness is the wide variety of approaches that may be taken in defining what it means to be a random sequence. The equivalence of resulting definitions may be regarded as providing evidence of a natural and robust theory. One particular approach concerns initial segment complexity. We let $2^{<\omega}$ denote the set of all finite binary strings. Given any $\sigma \in 2^{<\omega}$ we let $K(\sigma)$ denote the prefix-free complexity of σ—in other words, the length of the shortest string which produces σ via a fixed universal prefix-free machine. As always we identify sets of natural numbers and their characteristic sequences and for any sequence A and $n \in \omega$, we let $A \upharpoonright n$ denote the initial segment of A of length n. A sequence A is Martin-Löf random if the initial segments of A have close to maximal complexity. More precisely, A is Martin-Löf random if there exists a constant c such that $(\forall n)[K(A \upharpoonright n) \geq n - c]$. At the other end of the spectrum we say that A is K-trivial if there exists a constant c such that $(\forall n)[K(A \upharpoonright n) \leq K(n) + c]$ so that the initial segments of A have the smallest possible complexity (and where we identify each $\sigma \in 2^{<\omega}$ with the natural number n such that the binary representation of $n+1$ is 1σ).

Much recent research has been concerned with what can be said about the Turing degrees of Martin-Löf random or K-trivial sets. In his talk at a meeting in Cordoba in 2004, Kučera drew attention in particular to the issues surrounding Martin Löf (ML) cuppability.

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Definition 1.1. We let $A \oplus B$ denote the infinite sequence C such that for all n, $C(2n) = A(n)$ and $C(2n+1) = B(n)$. A set A is weakly ML-cuppable if $A \oplus Z \geq_T \emptyset'$ for some ML-random set $Z \not\geq_T \emptyset'$. A is ML-cuppable if one can choose $Z <_T \emptyset'$.

He raised the question as to which Δ^0_2 sets are (weakly) ML-cuppable, and whether one of these notions is sufficient to characterize K-triviality. Certainly if a computably enumerable (c.e.) set A is not ML-cuppable then it is K-trivial and Nies has shown that there exist non-computable c.e. sets which are not ML-cuppable.

Theorem 1.1. (Nies [AN]) Let $Y \in \Delta^0_2$ be Martin-Löf random. There exists a non-computable c.e. (in fact promptly simple) set A such that for each Martin-Löf random set Z:

$$Y \leq_T A \oplus Z \Rightarrow Y \leq_T Z.$$

Barmpalias [GB] has shown that in the statement of theorem 1.1 we may remove the condition that Y should be Martin-Löf random.

We call a degree **a** Martin-Löf random if it contains a set which is Martin-Löf random. Let us say that a degree **a** satisfies the cupping property if for every $\mathbf{b} > \mathbf{a}$ there exists $\mathbf{c} < \mathbf{b}$ with $\mathbf{b} = \mathbf{a} \vee \mathbf{c}$. Along the same lines as those issues discussed above, it seems a very natural question to ask; do all Martin-Löf random degrees satisfy the cupping property? The main result of this paper gives a negative answer to this question.

Definition 1.1. A degree **b** is a strong minimal cover for **a** if the degrees strictly below ***b*** are precisely the degrees below and including ***a***.

Theorem 1.2. There exists a Martin-Löf random degree which has a strong minimal cover.

This paper may also be seen as a continuation of the investigation carried out in [AL] as regards an old question of Yates'.

Question 1.1. (Yates) Does every minimal degree have a strong minimal cover?

Recall that $A \subseteq \omega$ is of hyperimmune-free degree if for every $f \leq_T A$ there exists a computable function h which majorizes f (i.e. such that $h(n) \geq f(n)$ for all n) and that $A \subseteq \omega$ is of fixed point free (FPF) degree if there exists $f \leq_T A$ such that $\phi_n \neq \phi_{f(n)}$ for all n, where ϕ_n is the n^{th} partial computable function according to some fixed effective listing. A degree is PA if it is the degree of a complete extension of Peano Arithmetic.

Theorem 1.3. (Lewis [AL]) All hyperimmune-free degrees which are not FPF have a strong minimal cover.

Given the similarity between the standard constructions of sets of minimal and of hyperimmune-free degree, perhaps the strongest result on the negative side of Yates' question is the result of Kučera's that the PA degrees satisfy the cupping property [AK], implying that there exists a hyperimmune-free degree with no strong minimal cover. Since all PA degrees are FPF, this theorem combines with theorem 1.3 to suggest the following question.

Question 1.2. Do all FPF degrees satisfy the cupping property, or do they at least fail to have a strong minimal cover?

A positive answer would have been very exciting. Given Kumabe's construction [MK] of a FPF minimal degree we would then have a negative solution to Yates' question together with a complete characterization of the hyperimmune-free degrees which have a strong minimal cover as those which are not FPF. Since every random degree is FPF theorem 1.2 provides a negative solution to question 1.2.

In what follows all notation and terminology will be standard unless explicitly stated otherwise. For background in computability theory we refer the reader to [BC] and [RS].

2. The proof of theorem 1.2

We consider the Cantor space 2^ω and denote the standard measure on 2^ω by μ. $\Lambda \subseteq 2^{<\omega}$ is said to be downward closed if, whenever $\tau \in \Lambda$, all initial segments of τ are in this set. Given any $\Lambda \subseteq 2^{<\omega}$ we denote by $[\Lambda]$ the set of infinite paths through Λ i.e. those sets A such that there are an infinite number of initial segments of A in Λ. $\mathcal{P} \subseteq 2^\omega$ is a Π^0_1 class if there exists a downward closed computable $\Lambda \subseteq 2^{<\omega}$ such that $\mathcal{P} = [\Lambda]$. We let $\mathcal{R}_e$ denote the e^{th} Π^0_1 class i.e. those sets which do not extend any string in W_e—and where the elements of W_e are thought of as elements of $2^{<\omega}$ according to the effective bijection $\omega \rightarrow 2^{<\omega}$ described previously. We let χ be an effective bijection from ω to the finite subsets of $2^{<\omega}$ and we write λ in order to denote the string of length 0. Generally we shall use the variable T to range over subsets of $2^{<\omega}$ which may not be downward closed. Given any $T \subseteq 2^{<\omega}$ and $\tau, \tau' \in T$ we say that τ' is a successor of τ in T if $\tau \subset \tau'$ and there does not exist $\tau'' \in T$ with $\tau \subset \tau'' \subset \tau'$. Non-empty T is said to be 2-branching if each $\tau \in T$ has precisely two successors in T. The strings of level n in T are those strings in T which have precisely n proper initial segments in T. We say that A computes T via Ψ if for every n the strings of level n in T are precisely the set $\chi(\Psi(A;n))$. Since we shall be interested in those Turing functionals which compute 2-branching T, we let $\{\Psi_e\}_{e\in\omega}$ be an effective listing of all those Turing functionals Ψ which satisfy the following conditions:

(1) for all $\sigma \in 2^{<\omega}$, $\Psi(\sigma;0) = m$ such that $\chi(m) = \{\lambda\}$;
(2) for $n > 0$, $\Psi(\sigma;n)$ is defined only if this computation converges in $< |\sigma|$ steps, and $\Psi(\sigma;n') \downarrow$ for all $n' < n$;
(3) for $n > 0$, if $\Psi(\sigma;n) \downarrow$ then $\chi(\Psi(\sigma;n))$ is a set of 2^n pairwise incompatible strings such that precisely two of these strings extend each member of $\chi(\Psi(\sigma;n-1))$.

We let $\{\Phi_e\}_{e\in\omega}$ be an effective listing of all Turing functionals Φ which satisfy the condition that for all $\sigma \in 2^{<\omega}$ and $n \in \omega$, $\Phi(\sigma;n)$ is defined only if this computation converges in $< |\sigma|$ steps, and $\Phi(\sigma;n') \downarrow$ for all $n' < n$.

2.1. The basic framework.

Since there exists a Π^0_1 class of positive measure which contains only Martin-Löf random sets it suffices to show that every Π^0_1 class of positive measure contains a set of degree which has a strong minimal cover. In fact the latter statement can be seen to be equivalent to theorem 1.2, since Kučera [AK] has shown that every Π^0_1 class of positive measure contains a set of every random degree. In [AL] it was observed that in order that A should be of degree with strong minimal cover it suffices that this set should satisfy the following condition;

(†) whenever A computes 2-branching T it computes some 2-branching $T' \subseteq T$ such that every $B \in [T']$ computes A.

In order that this paper should be as self-contained as possible we describe that proof again here. Let us consider, then, a straightforward approach to be taken in attempting to construct a strong minimal cover for any given degree. In doing so, lemma 2.1 will be useful.

Definition 2.1. We say that τ is $A\oplus$-compatible if, for all n such that $\tau(2n)\downarrow$, we have $\tau(2n) = A(n)$. Two finite strings τ, τ' are Φ-splitting if $\Phi(\tau)$ and $\Phi(\tau')$ are incompatible. We say that $T \subseteq 2^{<\omega}$ is Φ-splitting if whenever $\tau, \tau' \in T$ are incompatible, these two strings are Φ-splitting. We say that $T \subseteq 2^{<\omega}$ is a tree if it has a single element of level 0.

Lemma 2.1. Suppose Φ satisfies the condition that for all $\tau \in 2^{<\omega}$ and $n \in \omega$, $\Phi(\tau; n)$ is defined only if this computation converges in $< |\tau|$ steps, and $\Phi(\tau; n')\downarrow$ for all $n' < n$. If T_0 is an A-computable, 2-branching and Φ-splitting tree, then $T_1 = \{\Phi(\tau) : \tau \in T_0\}$ is an A-computable and 2-branching tree. Let T_2 be an A-computable and 2-branching subtree of T_1. Then $T_3 = \{\tau \in T_0 : \Phi(\tau) \in T_2\}$ is an A-computable and 2-branching subtree of T_0.

Proof. The proof is not difficult and is left to the reader. □

So now let us suppose that we wish to construct a strong minimal cover for $deg(A)$. In order to do so we must construct $B \geq_T A$ and satisfy all requirements:

$\mathcal{R}_i$: $\Phi_i(B)$ total $\Rightarrow (\Phi_i(B) \leq_T A$ or $B \leq_T \Phi_i(B))$;
$\mathcal{P}_i$: $B \neq \Phi_i(A)$.

In order to ensure that $B \geq_T A$ we can simply insist that B should be an $A\oplus$-compatible string. Thus we begin with the restriction that B should lie on the tree T_0 containing all strings of even length which are $A\oplus$-compatible, an A-computable 2-branching tree.

In order to meet all other requirements we define a sequence of finite strings $\{B_s\}_{s\in\omega}$ and a sequence of trees $\{T_s\}_{s\in\omega}$, so that ultimately we may define $B = \bigcup_s B_s$. We define B_0 to be the empty string. Suppose that by the end of stage s we have defined $B_s \in 2^{<\omega}$ on T_s, which is an A-computable 2-branching tree, in such a way that if B extends B_s and lies on T_s then all requirements $\mathcal{R}_i, \mathcal{P}_i$ for $i < s$ will be satisfied. At stage $s+1$ we might proceed, initially, just as if we were only trying to construct a minimal cover for A. We ask the question, "does there exist $\tau \supseteq B_s$ on T_s such that no two strings on T_s extending τ are Φ_s-splitting?".

If so: then let τ be such a string. We can define $T_{s+1} = T_s$ and (just to make the satisfaction of $\mathcal{P}_s$ explicit) define B_{s+1} to be some extension of τ on T_s sufficient to ensure $\mathcal{P}_s$ is satisfied.

If not: then we can define T'_s to be an A-computable 2-branching Φ_s-splitting subtree of T_s having B_s as the single string of level 0—the idea being that we shall eventually define T_{s+1} to be some subtree of T'_s. If B lies on T'_s then we shall have that $B \leq_T \Phi_s(B) \oplus A$. Of course this does not suffice, since for the satisfaction of $\mathcal{R}_s$ we require that $B \leq_T \Phi_s(B)$. Suppose, however, that we know A satisfies the property (†). Lemma 2.1 then suffices to ensure that we can define T_{s+1} to be a subtree of T'_s which is an A-computable 2-branching tree, and which satisfies the property that if $B \in [T_{s+1}]$ then $A \leq_T \Phi_s(B)$ so that, since $B \leq_T \Phi_s(B) \oplus A$, $B \leq_T \Phi_s(B)$. Then we can define B_{s+1} to be an extension of B_s lying on T_{s+1} sufficient to ensure the satisfaction of $\mathcal{P}_s$.

In order to prove theorem 1.2, then, our basic framework is as follows. We suppose that we are given downward closed and computable $\Lambda \subseteq 2^{<\omega}$ which defines a Π^0_1 class $\mathcal{P} = [\Lambda]$ of positive measure. We show that there exists $A \in \mathcal{P}$ which satisfies the requirements:

$\mathcal{Q}_j$: if A computes 2-branching T via Ψ_j then A computes some 2-branching $T' \subseteq T$ such that every $B \in [T']$ computes A.

Initially we define $\mathcal{P}_0 = \mathcal{P}$ and $\Lambda_0 = \Lambda$. For each j, given $\mathcal{P}_j$ of positive measure such that every set in $\mathcal{P}_j$ satisfies the requirements $\mathcal{Q}_{j'}$ for $j' < j$, we define $\mathcal{P}_{j+1} \subseteq \mathcal{P}_j$ of positive measure such that every set in $\mathcal{P}_{j+1}$ satisfies requirement $\mathcal{Q}_j$. Since the Cantor space is compact $\bigcap_j \mathcal{P}_j$ will then be non-empty.

2.2. **Defining $\mathcal{P}_{j+1}$.** Suppose that we are given $\mathcal{P}_j = [\Lambda_j]$ which is of positive measure and such that Λ_j is downward closed and computable, and let us consider how to define $\mathcal{P}_{j+1} = [\Lambda_{j+1}]$.

The first possibility we have to consider in defining $\mathcal{P}_{j+1}$ is that, for some $A \in \mathcal{P}_j$, it may be the case that $\Psi_j(A)$ is partial. It is a familiar technique in dealing with Π^0_1 classes of positive measure, see for example [AK] or [DM], that we may take a Π^0_1 subclass $\mathcal{P}'_j \subseteq \mathcal{P}_j$ which is of positive measure and such that the intersection of $\mathcal{P}'_j$ with any other Π^0_1 class is either empty or is of positive measure.

Lemma 2.2. (Kučera [AK]) Let $\mathcal{P}$ be a Π^0_1 class such that $\mu(\mathcal{P}) > 0$. Then there exists a Π^0_1 subclass $\mathcal{P}' \subseteq \mathcal{P}$ and a computable function $g : \omega \to \omega$ such that $\mu(\mathcal{P}') > 0$ and:

$$(\forall e)[\mathcal{P}' \cap \mathcal{R}_e \neq \emptyset \Rightarrow \mu(\mathcal{P}' \cap \mathcal{R}_e) > 2^{-g(e)}].$$

A particularly simple proof of lemma 2.2 is provided in [DM]. In order to deal with the potential partiality of Ψ_j we begin by supposing given $\mathcal{P}'_j \subseteq \mathcal{P}_j$ as provided by lemma 2.2 such that $\mathcal{P}'_j = [\Lambda'_j]$ for some downward closed computable Λ'_j, and we then proceed to ask the following question:

($\star$) Does there exist $A \in \mathcal{P}'_j$ and $n \in \omega$ such that $\Psi_j(A; n) \uparrow$?

In the case that ($\star$) receives a positive answer we can just define Λ_{j+1} to be the set of all $\sigma \in \Lambda'_j$ such that $\Psi_j(\sigma; n) \uparrow$. Since $\mathcal{P}_{j+1} = [\Lambda_{j+1}]$ has non-empty intersection with $\mathcal{P}'_j$ it is of positive measure.

So let us suppose that ($\star$) is answered in the negative. Thus $\Psi_j(A)$ is total for all $A \in \mathcal{P}'_j$. The next thing we do is to perform a certain 'tidying' operation. The problem at the moment is that if T is computed by $A \in \mathcal{P}'_j$ via Ψ_j, then different strings of the same level in T may be of different lengths. Of course, different strings of the same level but in different T (corresponding to different $A \in \mathcal{P}'_j$) may also be of different lengths, and for any n the shortest $\sigma \subset A$ such that $\Psi_j(\sigma; n)\downarrow$ may vary with different values of A. The argument that follows will be much simpler if we can remove these complications. Thus we define two computable functions h, l and a Turing functional Ξ_0, and the point in doing so is to replace each T which is computed by $A \in \mathcal{P}'_j$ via Ψ_j, with 2-branching T' which A computes via Ξ_0 and such that the downward closure of T' is a subset of the downward closure of T. We define Ξ_0 in such a way that all the various T' will satisfy the convenient property that the strings of level n in T' are of length $l(n)$. The role of $h(n)$ is to specify the length of the strings σ on which we define $\Xi_0(\sigma; n)$. Since it is not difficult to see that we can define these functions, the particular details of how we do so are not important. We give the details for the sake of completeness.

Ξ_0, h and l are defined by recursion. We define $h(0) = l(0) = 0$ and $\Xi_0(\lambda; 0) = \chi^{-1}(\{\lambda\})$. Now suppose we are given $h(n)$, $l(n)$ and that $\Xi_0(\sigma; n)$ is defined for all $\sigma \in \Lambda'_j$ of length $h(n)$. Let m be greater than $l(n)$ and find $m' > h(n)$ such that for every $\sigma \in \Lambda'_j$ of length m', $\Psi_j(\sigma; m)\downarrow$. The point so far is just this; for every $\sigma \in \Lambda'_j$ of length m' and every $\tau \in \chi(\Xi_0(\sigma; n))$ there must be at least two incompatible strings in $\chi(\Psi_j(\sigma; m))$ which extend τ. The remaining problem is that these strings may split below many different lengths. So define $l(n+1)$ to be the length of the longest string in any of the sets $\chi(\Psi_j(\sigma; m))$ for $\sigma \in \Lambda'_j$ of length m'. Find $m'' > m'$ such that for every $\sigma \in \Lambda'_j$ of length m'', $\Psi_j(\sigma; l(n+1))\downarrow$—the point being that all the strings in $\chi(\Psi_j(\sigma; l(n+1)))$ must be of length at least $l(n+1)$. Define $h(n+1) = m''$ and for each $\sigma \in \Lambda'_j$ of length m'' define $\Xi_0(\sigma; n+1)$ to be equal to some d such that $\chi(d)$ is a set of 2^{n+1} pairwise incompatible strings of the form $\tau \restriction l(n+1)$ for $\tau \in \chi(\Psi_j(\sigma; l(n+1)))$, with precisely two strings extending each member of $\chi(\Xi_0(\sigma; n))$.

In order to ensure that every $A \in \mathcal{P}_{j+1}$ satisfies requirement $\mathcal{Q}_j$ we shall construct Φ and Ξ_1 such that if $A \in \mathcal{P}_{j+1}$ computes T' via Ξ_0 then A computes 2-branching $T'' \subseteq T'$ via Ξ_1 and for every $B \in [T'']$ we have $\Phi(B) = A$. We let k be a computable function such that $\Sigma_{n>0} 2^{-k(n)} < \mu(\mathcal{P}'_j)$. We shall describe a computable process such that at each step in the process various clopen sets of strings are removed from $\mathcal{P}'_j$ in order to define $\mathcal{P}_{j+1}$. At step $n \geq 1$ we shall remove measure at most $2^{-k(n)}$ from $\mathcal{P}'_j$, so that $\mathcal{P}_{j+1}$ which is the set of strings in $\mathcal{P}'_j$ which are not removed at any step of the process, will be of positive measure. The point of this process is to define Ξ_1 and Φ, and the strings we remove from $\mathcal{P}'_j$ are those strings on which we are not able to define these values in an appropriate way. At each step n in the process we shall look to define the strings of level n in T'' which is computed via Ξ_1, and for each string τ of level n in T'' we shall look to define $\Phi(\tau; n-1)$.

First we define $\Xi_1(\lambda; 0) = \chi^{-1}(\{\lambda\})$. At step 1, in order to define Φ on argument 0 and Ξ_1 on argument 1, we shall choose some large m and then enumerate axioms of the form $\Phi(\tau; 0) = i \in \{0, 1\}$ for τ in $\chi(\Xi_0(\sigma; m))$ with σ of length $h(m)$. If for each σ of length $h(m)$ there exist sufficiently many corresponding τ (i.e. $\tau \in \chi(\Xi_0(\sigma; m))$) on which we can enumerate axioms for Φ because we have chosen m which is sufficiently large, then we shall be able to ensure that the proportion of the σ of length $h(m)$ in Λ'_j for which there exist at least two corresponding τ for which we enumerate the axiom $\Phi(\tau; 0) = \sigma(0)$ is greater than $1 - 2^{-k(1)}$. Thus the measure of those strings which extend some σ of length $h(m)$ in Λ'_j for which there *do not* exist at least two corresponding τ for which we enumerate the axiom $\Phi(\tau; 0) = \sigma(0)$ is less than $2^{-k(1)}$. At this point we remove all strings extending such σ from $\mathcal{P}'_j$ and in doing so we remove measure at most $2^{-k(1)}$. Let Π_0 be the set of strings in Λ'_j of length $h(m)$ which remain. For each σ in Π_0 we can define $\Xi_1(\sigma; 1)$ to be (the code for) a set of two strings in $\chi(\Xi_0(\sigma; m))$, each of which correctly computes $\sigma(0)$ via Φ. For all those strings σ of length $h(m)$ that remain, then, we have successfully defined T'' which σ computes via Ξ_1 up to level 1 and for all strings τ of level 1 in T'', $\Phi(\tau)$ correctly computes σ on argument 0.

We then proceed to define Φ on argument 1, Ξ_1 on argument 2 and Π_1. We choose m' sufficiently large and then enumerate axioms of the form $\Phi(\tau; 1) = i \in \{0, 1\}$ for τ in $\chi(\Xi_0(\sigma; m'))$ with σ of length $h(m')$ and extending some string in Π_0. If for each σ of length $h(m')$ there exist sufficiently many corresponding τ, then we shall be able to ensure that the proportion of the σ of length $h(m')$ extending some string in Π_0 such that for each string τ in $\chi(\Xi_1(\sigma; 1))$ there exist at least two strings $\tau' \supset \tau$ in $\chi(\Xi_0(\sigma; m'))$ for which we enumerate the axiom $\Phi(\tau'; 1) = \sigma(1)$, is greater than $1 - 2^{-k(2)}$. So let Π_1 be the set of those σ of length $h(m')$ for which this condition holds. We can then remove all those elements of $\mathcal{P}'_j$ which do not extend a string in Π_1 and the measure of the elements removed will be less than $2^{-k(2)}$. For the strings σ in Π_1 we can define $\Xi_1(\sigma; 2)$ to be (the code for) a set of four pairwise incompatible strings each of which correctly computes $\sigma(1)$ via Φ, and such that precisely two of these strings extend each element of $\chi(\Xi_1(\sigma; 1))$. Then we proceed by induction in the obvious way.

Hopefully it is clear, then, why we must insist that each $\mathcal{P}_j$ is of positive measure. The argument outlined above relies on the fact that if we remove a set of small measure from $\mathcal{P}'_j$ at each step of the process then $\mathcal{P}_{j+1}$ will be non-empty.

Lemma 2.3. Let X and Y be finite sets and suppose that $m_2 > m_1 + 1$. Suppose that for each $y < 2^{m_1}$ and for each $a \in X$, $f_y(a)$ is a subset of Y of size 2^{m_2}, and that to each element of X is associated a 'colour' either 0 or 1. Then we may colour each element of Y either 0 or 1 so that the proportion of $a \in X$ for which it is the case that for every $y < 2^{m_1}$ there exist at least 2 elements of $f_y(a)$ which are the same colour as a is at least:

$$1 - \frac{2^{m_1+1}}{2^{m_2-1}}.$$

Proof. Before proving the lemma let us consider its meaning. In the first step of the procedure outlined above we looked to define Ξ_1 on argument 1 and Φ on argument 0. With this in mind, consider the statement of the lemma in the case that $m_1 = 0$. Think of m_2 as being the value m used in the description of this first step of the procedure. Think of X as the set of strings $\sigma \in \Lambda'_j$ of length $h(m)$. For each $\sigma \in X$ think of $f_0(\sigma)$ as being $\chi(\Xi_0(\sigma; m))$. Think of Y as the union of all sets of strings $\chi(\Xi_0(\sigma; m))$ such that σ is in X. The colour given to an element of X should be thought of as the value $\sigma(0)$. The colour given to elements of Y corresponds to the axioms we enumerate for Φ. In this case the lemma tells us precisely what we need to know—that if we choose m which is sufficiently large, then we shall be able to ensure that the proportion of the σ of length $h(m)$ in Λ'_j for which there exist at least two $\tau \in \chi((\Xi_0(\sigma; m))$ for which we enumerate the axiom $\Phi(\tau; 0) = \sigma(0)$ is greater than $1 - 2^{-k(1)}$.

Now consider step $n + 1$ of the procedure and put $m_1 = n$. We have already defined Π_{n-1} and for each string $\sigma \in \Pi_{n-1}$, $\chi(\Xi_1(\sigma; n))$ is a set of 2^n pairwise incompatible strings. We wish to choose some large m' and enumerate axioms for Ξ_1 on the strings in Λ'_j which extend some string in Π_{n-1} and which are of length $h(m')$. Let $h(m)$ be the length of the strings in Π_{n-1} and think of m_2 as $m' - m$. Think of X as the set of strings $\sigma \in \Lambda'_j$ of length $h(m')$ which extend some string in Π_{n-1}. For each $y < 2^n$ and each $\sigma \in X$ think of $f_y(\sigma)$ as the set of strings in $\chi(\Xi_0(\sigma; m'))$ which extend the y^{th} element of $\chi(\Xi_1(\sigma; n))$ (ordered from left to right, say). Think of Y as the union of all these sets $f_y(\sigma)$ such that $\sigma \in X$ and $y < 2^n$. The colour given to an element σ of X now corresponds to the value $\sigma(n)$. Once again the lemma now tells us precisely what we need to know—if we choose m' large enough then we shall be able to ensure that the proportion of the σ of length $h(m')$ in Λ'_j extending some string in Π_{n-1} such that for each string τ in $\chi(\Xi_1(\sigma; n))$ there exist at least two strings $\tau' \supset \tau$ in $\chi(\Xi_0(\sigma; m'))$ for which we enumerate the axiom $\Phi(\tau'; n) = \sigma(n)$, is greater than $1 - 2^{-k(n+1)}$.

In order to prove the lemma, we first we prove something simpler. Let X and Y be finite sets such that to each element of X is associated m elements in Y and a 'colour' either 0 or 1. Then we may colour each element of Y either 0 or 1 in such a way that the proportion of those elements a of X for which there exists at least one associated element of Y which is the same colour as a, is at least $m/(m + 1)$.

In order to show that this is the case we describe a method for colouring the elements of Y which we shall refer to as the 'standard method'.

Stage 0. Define $X_0 = X$ and $Y_0 = Y$.

Stage $s + 1$. For each element b of Y_s and each $i \in \{0, 1\}$ let $z_i(b)$ be the number of elements of X_s associated with b and which are coloured i. Choose any $b \in Y_s$, let i be such that $z_i(b)$ is largest (and just choose i if $z_0(b) = z_1(b)$) and give b the colour i. Define $Y_{s+1} = Y - \{b\}$ and define

X_{s+1} to be the elements a of X for which is not yet the case that there exists at least one associated element of Y which is the same colour as a.

In order to see that this procedure does what it is supposed to we consider two counters YIN and $YANG$. Initially $YIN = YANG = 0$. Whenever we give a colour i to an element b of Y we increase $YANG$ by $z_i(b)$ and we increase YIN by $z_{1-i}(b)$. Let YIN and $YANG$ take their final values. The number of elements a of X for which there exists at least one associated element of Y which is the same colour as a is precisely the final value $YANG$. For each element a of X for which there is not an associated member of Y of the same colour, we must increase the value YIN by m, in order to rule out correctly colouring each of the m elements of Y with which a is associated. Thus the number of elements a of X for which there does not exist at least one associated element of Y which is the same colour as a is at most YIN/m. Since $YIN \leq YANG$ the result follows.

Now suppose given X and Y as in the statement of the lemma. We describe a method for colouring Y which we call the 'special method'. First we form a set X' by replacing each element a of X with 2^{m_1+1} distinct elements which are given the same colour as a and which are divided into 2^{m_1} pairs. The first element of the y^{th} pair for each $y < 2^{m_1}$, we associate with 2^{m_2-1} elements of $f_y(a)$ and the second element of the pair we associate with the remaining elements of $f_y(a)$. Then we colour Y according to the standard method for X' and Y.

Let's say that the colouring is good for $a \in X'$ if there exists at least one associated member of Y with the same colour as a. The total number of elements in X' is $2^{m_1+1}\,|X|$. Since each element of X' is associated with 2^{m_2-1} elements of Y it follows from the argument above that the total number of elements a of X' such that the colouring is not good for a is at most:

$$\frac{2^{m_1+1}\,|X|}{2^{m_2-1}+1}$$

The elements of X' are divided into $2^{m_1}\,|X|$ pairs. Let us say that the colouring is good for a pair if it is good for both elements of the pair. Now the total number of pairs for which the colouring is not good can be at most the number of $a \in X'$ such that the colouring is not good for a. Let us say that the colouring is good for $a \in X$ if the colouring is good for every one of the 2^{m_1} pairs which replaced a in forming X'. The total number of $a \in X$ for which the colouring is not good can be at most the number of $a \in X'$ such that the colouring is not good for a. Thus the proportion of $a \in X$ for which the colouring is not good is at most:

$$\frac{2^{m_1+1}}{2^{m_2-1}+1}$$

The proportion of $a \in X$ for which the colouring is good is therefore at least:

$$1 - \frac{2^{m_1+1}}{2^{m_2-1}+1} \geq 1 - \frac{2^{m_1+1}}{2^{m_2-1}}$$

□

The formal definition of Φ and $\mathcal{P}_{j+1}$. First we define $\Xi_1(\lambda;0)=\chi^{-1}(\{\lambda\})$. In order to define Φ on argument 0 and Ξ_1 on argument 1, we simply choose m such that $1-(2/2^{m-1})>1-2^{-k(1)}$. We define X to be the strings σ in Λ'_j of length $h(m)$. We define $m_1=0$, $m_2=m$. We define $f_0(\sigma)$ for each $\sigma\in X$ to be the set of strings $\chi(\Xi_0(\sigma;m))$ and we define $Y=\bigcup_{\sigma\in X}f_0(\sigma)$. To each element σ of X we associate the colour $\sigma(0)$ and we use the special method in order to colour Y. Whenever $\tau\in Y$ is coloured with $i\in\{0,1\}$ we enumerate the axiom $\Phi(\tau;0)=i$. At this point we also make the following definitions. We define Π_0 to be the $\sigma\in\Lambda'_j$ of length $h(m)$ for which there exist at least two $\tau\in\chi(\Xi_0(\sigma;m))$ with $\Phi(\tau;0)=\sigma(0)$ and for each such σ we choose two such τ, let's call them τ_0 and τ_1, and we define $\Xi_1(\sigma;1)=\chi^{-1}(\{\tau_0,\tau_1\})$. We define the strings in Λ_{j+1} of length $\leq h(m)$ to be those strings which are initial segments of some $\sigma\in\Pi_0$ and we define $d(0)=m$—here d is just an auxiliary function which is useful. Note that in defining $\mathcal{P}_{j+1}=[\Lambda_{j+1}]$ from $\mathcal{P}'_j$ we have so far removed a set of measure at most $2^{-k(1)}$.

Now let us suppose that we wish to define Φ on argument n, and that we have defined Π_{n-1}, $\Xi_1(\sigma;n)$ for all $\sigma\in\Pi_{n-1}$ and $d(n-1)$. In order to define Φ on argument n we put $m_1=n$, and we choose m_2 sufficiently large such that:

$$1-\frac{2^{m_1+1}}{2^{m_2-1}}>1-2^{-k(n+1)}.$$

Putting $m'=d(n-1)+m_2$ we define X to be the strings σ in Λ'_j which extend a string in Π_{n-1} and which are of length $h(m')$. For each σ in X and for each $y<2^n$ we define $f_y(\sigma)$ to be the set of strings in $\chi(\Xi_0(\sigma;m'))$ which extend the y^{th} element of $\chi(\Xi_1(\sigma;n))$ (ordered from left to right say) and we define $Y=\bigcup_{\sigma,y}f_y(\sigma)$. To each element σ of X we associate the colour $\sigma(n)$ and we use the special method in order to colour Y. Whenever $\tau\in Y$ is coloured with $i\in\{0,1\}$ we enumerate the axiom $\Phi(\tau;n)=i$. We define Π_n to be the $\sigma\in\Lambda'_j$ of length $h(m')$ extending some string in Π_{n-1} and for which it is the case for every $\tau\in\chi(\Xi_1(\sigma;n))$ that there exist at least two incompatible extensions $\tau'\in\chi(\Xi_0(\sigma;m'))$ with $\Phi(\tau';n)=\sigma(n)$, and for each such σ we choose two such τ' for each $\tau\in\chi(\Xi_1(\sigma;n))$ and we define $\Xi_1(\sigma;n+1)$ to be equal to x such that $\chi(x)$ is this set of 2^{n+1} strings. We define the strings in Λ_{j+1} of length l with $h(d(n-1))<l\leq h(m')$ to be those strings which are initial segments of some $\sigma\in\Pi_n$ and we define $d(n)=m'$. Note that in defining Φ on argument n we have removed a set of measure at most $2^{-k(n+1)}$ from $\mathcal{P}'_j$.

There is just one small task which remains. The (very minor) problem at the moment is that if $A\in\mathcal{P}_{j+1}$ computes T via Ψ_j and computes T'' via Ξ_1 then every set in T'' computes A and $[T'']\subseteq[T]$ but we do not necessarily have that $T''\subseteq T$. This is easily remedied. We define an appropriate functional Ξ_2 which will compute the tree required. Given an oracle for A this is how one may compute $\Xi_2(A)$. We have $\Xi_2(A;0)=\chi^{-1}(\{\lambda\})$. Once we have defined $\Xi_2(A;n)$, in order to define $\Xi_2(A;n+1)$ we enumerate T'' and T until for each τ in $\Xi_2(A;n)$ we see two incompatible strings τ' in T

which extend τ and which are extended by strings in T''. Let Π be this set of τ' and define $\Xi_2(A; n+1) = \chi^{-1}(\Pi)$.

3. Final remarks.

The forcing argument described here can easily be combined with the standard technique for constructing A of hyperimmune-free degree. Suppose that we wish to act in order to ensure that either $\Phi_e(A)$ is not total or is majorized by a computable function. Given $\mathcal{P}_j$ of positive measure we take $\mathcal{P}'_j = [\Lambda'_j]$ as provided by lemma 2.2. Then we ask; does there exist $A \in \mathcal{P}'_j$ and $n \in \omega$ such that $\Phi_e(A; n) \uparrow$? In the case that this question receives a positive answer we can just define Λ_{j+1} to be the set of all $\sigma \in \Lambda'_j$ such that $\Phi_e(\sigma; n) \uparrow$. Since $\mathcal{P}_{j+1} = [\Lambda_{j+1}]$ has non-empty intersection with $\mathcal{P}'_j$ it is of positive measure. If this question receives a negative response then the standard argument suffices to show that for any $A \in \mathcal{P}'_j$ there exists a computable function which majorizes $\Phi_e(A)$. In this case we may therefore define $\mathcal{P}_{j+1} = \mathcal{P}'_j$. We therefore have:

Theorem 3.1. There exists a hyperimmune-free degree which is random and which has a strong minimal cover. In particular, the hyperimmune-free degrees with strong minimal cover cannot be characterized as those which are not FPF.

References

[GB] G. BARMPALIAS, 'Random non-cupping revisited', *Journal of Complexity*, 22 (2006), 850-857.

[BC] S.B. COOPER, *Computability theory*, Chapman & Hall, CRC Press, Boca Raton, FL, New York, London (2004).

[DM] R. DOWNEY and J. MILLER, 'A basis theorem for Π^0_1 classes of positive measure and jump inversion for random reals', *Proceedings of the American Mathematical Society*, 134 (1) (2006), 283–288.

[AK] A. KUČERA, 'Measure, Π^0_1-classes and complete extensions of PA', *Recursion Theory Week* (Oberwolfach, 1984), volume 1141 of Lecture Notes in Math. 245-259, Springer, Berlin.

[MK] M. KUMABE, 'A fixed point free minimal degree', (1993), unpublished.

[AL] A.E.M. LEWIS, 'Π^0_1 classes, strong minimal covers and hyperimmune-free degrees', (2006), submitted.

[AN] A. NIES, 'Non-cupping and randomness', *Proceedings of the American Mathematical Society*, 135 (3) (2007), 837-844.

[RS] R.I. SOARE, *Recursively enumerable sets and degrees*, Springer, New York, (1987).

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