

A NOTE ON THE JOIN PROPERTY

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ABSTRACT. A Turing degree \mathbf{a} satisfies the *join* property if, for every non-zero $\mathbf{b} < \mathbf{a}$, there exists $\mathbf{c} < \mathbf{a}$ with $\mathbf{b} \vee \mathbf{c} = \mathbf{a}$. It was observed in [N4] that all degrees which are non- GL_2 satisfy the join property. This, however, leaves open many questions. Do all a.n.r. degrees satisfy the join property? How about the PA degrees or the Martin-Löf random degrees? A degree \mathbf{b} satisfies the cupping property if, for every $\mathbf{a} > \mathbf{b}$, there exists $\mathbf{c} < \mathbf{a}$ with $\mathbf{b} \vee \mathbf{c} = \mathbf{a}$. Is satisfying the cupping property equivalent to all degrees above satisfying join? We answer all of these questions by showing that above every low degree there is a low degree which does not satisfy join. We show, in fact, that all low fixed point free degrees \mathbf{a} fail to satisfy join, and moreover, that the non-zero degree below \mathbf{a} without any joining partner can be chosen to be a c.e. degree.

1. INTRODUCTION

The issue as to which Turing degrees satisfy the join property has long been of interest to researchers in computability theory. Robinson showed in his PhD thesis of 1972 [RR] that $\mathbf{0}'$ satisfies the join property, and this result was subsequently improved, by stages, in a number of directions. Greenberg, Montalbán and Shore [GMS] have shown that actually all generalized high degrees satisfy the *complementation property*, where a degree \mathbf{a} is generalized high if $\mathbf{a}' = (\mathbf{a} \vee \mathbf{0}')$, and where \mathbf{a} satisfies the complementation property if, for every non-zero $\mathbf{b} < \mathbf{a}$, there exists \mathbf{c} with $\mathbf{b} \vee \mathbf{c} = \mathbf{a}$ and $\mathbf{b} \wedge \mathbf{c} = \mathbf{0}$. It was shown in [N4] that all degrees which are non- GL_2 satisfy the join property, where a degree \mathbf{a} is GL_n if $\mathbf{a}^{(n)} = (\mathbf{a} \vee \mathbf{0}')^{(n-1)}$ and is L_n if $\mathbf{a} < \mathbf{0}'$ and $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$. Since initial segment results, see for example [ML], suffice to show that there are degrees which are L_1 which do not satisfy the join property, and also degrees which are L_2 but not L_1 of this kind, the latter result is sharp in terms of the jump hierarchy.

Every degree which is not above $\mathbf{0}'$, however, is bounded by a degree which is GL_1 , and so the degrees which are non- GL_2 are not upward closed. In looking to understand for which degrees it is the case that all degrees in the upper cone satisfy join, it becomes natural to ask whether all PA degrees or all a.n.r. degrees satisfy join, because these are upward closed classes which have played significant roles in degree theory. A degree is PA if it contains a set which effectively codes a complete and consistent extension of Peano Arithmetic, and a degree is a.n.r. (array non-recursive) if it contains a function which is not dominated by the modulus of convergence function for some computable enumeration of the halting problem. Both of these classes have been shown to satisfy the cupping property [DJS], [AK]. Another natural question is as to whether satisfaction of the cupping property is

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equivalent to all degrees above satisfying join. The result of this paper answers all of these questions.

We let $\{\Psi_n\}_{n \in \omega}$ be an effective listing of all Turing functionals. A degree is *low* if it is L_1 . A degree is fixed point free if it contains a function f such that, for all n , $\Psi_n \neq \Psi_{f(n)}$. A function f is diagonally non-computable (d.n.c.) if, for all n , $f(n) \neq \Psi_n(\emptyset; n)$. It is well known that a degree is fixed point free iff it contains a d.n.c. function.

Theorem 1.1. *Every low fixed point free degree fails to satisfy the join property. More specifically, if \mathbf{a} is low and fixed point free, then there exists a promptly simple degree $\mathbf{b} < \mathbf{a}$ such that no degree $\mathbf{d} < \mathbf{a}$ joins \mathbf{b} above \mathbf{a} .*

Corollary 1.2. Above every low degree there is a low degree which does not satisfy the join property.

Proof. Given any low degree \mathbf{c} consider $\mathbf{a} > \mathbf{c}$ which is both low over \mathbf{c} and PA over it. Any degree which is PA over \mathbf{c} is PA, and therefore fixed point free. \square

Corollary 1.3. There exist a.n.r. degrees which do not satisfy the join property.

Proof. There are a.n.r. degrees which are low, and the a.n.r. degrees form an upward closed class. \square

Corollary 1.4. There exist PA degrees which do not satisfy the join property.

Proof. There exist PA degrees which are low, and all PA degrees are fixed point free. \square

Corollary 1.5. There exist Martin-Löf random degrees which do not satisfy the join property.

Proof. As for Corollary 1.4. \square

Corollary 1.6. Satisfaction of the cupping property is not equivalent to all degrees above satisfying join.

Proof. Every PA degree strictly bounds a PA degree, and all PA degrees satisfy the cupping property. \square

This leaves open two significant questions:

Question 1.7. Can $\mathbf{0}'$ be defined as the least degree such that all degrees above it satisfy join?

Question 1.8. In $\mathcal{D}[\leq \mathbf{0}']$, can L_2 be defined as those degrees for which it is not the case that all degrees above satisfy join?

Our notation and terminology will be standard, and will follow [RS] unless explicitly stated otherwise. We shall use the variables i, j, k, n, m, s, t for elements of ω ; ϕ, ψ, τ, ζ for elements of $2^{<\omega}$; f, g, h for total functions $\omega^n \mapsto \omega$; v for partial functions $\omega \mapsto \omega$; A, B, C, D for subsets of ω . We let W_i denote the i th c.e. set. $\Psi(A; n)$ denotes the output of the Turing functional Ψ on argument n , given oracle input A . We shall assume that, for any ϕ, i, n, m , $\Psi_i(\phi; n) \downarrow = m$ only if this computation converges in less than $|\phi|/2$ steps and $\Psi_i(\phi; n') \downarrow$ for all $n' < n$ (we consider $|\phi|/2$ here because often we shall consider oracles of the form $A \oplus B$, where the \oplus function is defined subsequently). A c.e. set B is *promptly simple* if it has

infinite complement and a computable enumeration $\{B_s\}_{s \in \omega}$ such that whenever W_i is infinite, $\exists s \exists n [n \in W_{i,s} - W_{i,s-1} \ \& \ n \in B_s]$. We let $A \upharpoonright n$ denote the initial segment of A of length n . For (possibly) partial functions v_0, v_1 , we let $v_0 \oplus v_1$ be the (possibly) partial function v such that $v(2n) = v_0(n)$ if this value is defined and is undefined otherwise, $v(2n+1) = v_1(n)$ if this value is defined and is undefined otherwise. At any point during a construction, by a *large* number, we mean a number larger than any previously mentioned during the construction.

2. THE PROOF OF THEOREM 1.1

We consider first a proof of the result due to Slaman and Steel [SS], and independently due to Cooper [BC], that there exist non-zero c.e. degrees $\mathbf{b} < \mathbf{a}$ such that no Δ_2^0 degree $\mathbf{d} < \mathbf{a}$ joins \mathbf{b} above \mathbf{a} . Actually we shall build \mathbf{b} which is promptly simple, since there is really no extra complication involved in doing so. We construct c.e. sets B and C such that the following requirements are satisfied:

\mathcal{P}_i : If W_i is infinite then $\exists s \exists n [n \in W_{i,s} - W_{i,s-1} \ \& \ n \in B_s]$;

\mathcal{Q}_i : If $\Psi_i(B \oplus C)$ ($= D_i$ say) is total and $\Psi_i(B \oplus D_i) = C$ then $\Gamma_i(D_i) = B$;

where each Γ_i is a functional that we build during the course of the construction. B will have infinite complement by construction. Note that it is sufficient to consider only the functional Ψ_i for the requirement \mathcal{Q}_i , rather than considering $\Psi_j(B \oplus C) = D_i$ and $\Psi_k(B \oplus D_i)$, since as long as $D_i \neq C$ one Turing functional Ψ_i may simulate either Ψ_j or Ψ_k according to whether some given string, which is sufficient to distinguish $B \oplus C$ and $B \oplus D_i$, is an initial segment of the oracle. In what follows we will often abuse notation by writing D_i in order to denote $\Psi_i(B \oplus C)$, even though this may actually be a partial function.

We shall have one strategy for each requirement, and the requirements and their corresponding strategies are prioritized $\mathcal{P}_0, \mathcal{Q}_0, \mathcal{P}_1, \mathcal{Q}_1 \dots$. At each stage s of the construction we perform another step of the instructions for each of the first s strategies in turn. The construction will have finite injury. Since the strategies are very simple and the interactions between them are not complicated, it is probably easiest just to describe the strategies directly and then verify that they work.

In what follows we adopt the standard convention that when discussing any point in the construction, we may write B, C etc in order to denote their present values. At any point during the construction we let ζ_i denote the finite string $\Psi_i(B \oplus C)$ (which we can assume to be binary).

The \mathcal{Q}_i strategy. At any point in the construction the length of agreement for \mathcal{Q}_i is the greatest m such that $C \upharpoonright m \subseteq \Psi_i(B \oplus \zeta_i)$. At each stage at which it is run the strategy considers the least n such that $\Gamma_i(\zeta_i; n) \uparrow$ and if n is less than the length of agreement then it proceeds as follows. Let ϕ and ζ be, respectively, the initial segments of B and ζ_i used in the computation $\Psi_i(B \oplus \zeta_i; n) = C(n)$. If there exists a shortest ζ' with $\zeta \subseteq \zeta' \subseteq \zeta_i$ such that $\Gamma_i(\zeta'; n') \downarrow$ for all $n' < n$ and for which it is consistent with all axioms previously enumerated to enumerate the axiom $\Gamma_i(\zeta'; n) = \phi(n)$, then enumerate this axiom.

The \mathcal{P}_i strategy. When this strategy enumerates some n into B it must be able to ensure for each $\mathcal{Q}_{i'}$ of higher priority, that if $D_{i'}$ is total and $\Psi_{i'}(B \oplus D_{i'}) = C$, then $D_{i'}$ cannot extend any string ζ for which we have enumerated an axiom

$\Gamma_{i'}(\zeta; n) = 0$. Therefore the strategy will not enumerate n into B , unless there exists $\phi = B \upharpoonright n'$ for some $n' < n$ and m such that for all such ζ and i' , $\Psi_{i'}(\phi \oplus \zeta; m) = 0$. Upon enumerating n into B the strategy then enumerates m into C and looks to preserve ϕ as an initial segment of B . This latter task it attempts to achieve simply by initializing all lower priority strategies – these strategies will only be allowed to enumerate numbers into B or C which are greater than the last stage at which they were initialized. Any higher priority strategy $\mathcal{P}_{i''}$ may subsequently enumerate a number into B which is less than $|\phi|$, but then *this* strategy will similarly ensure that any such ζ cannot be an initial segment of $D_{i'}$ if it is to be the case that $D_{i'}$ is total and $\Psi_{i'}(B \oplus D_{i'}) = C$ (and if $i'' > i'$), since it will also enumerate some m' into C and preserve a (shorter) initial segment of B . The instructions for the strategy are as follows.

The instructions for \mathcal{P}_i . When the strategy is first run (subsequent to any initialization) it chooses a large agitator m and then a large marker $n > m$. At every subsequent stage s at which it has not yet been declared satisfied it performs the following steps:

- (1) Check to see whether there exists any $i' < i$ for which the strategy has not previously *seen convergence* (this will initially be all $i' < i$) and for which $\Gamma_{i'}(\zeta_{i'}; m) \downarrow$. If so then initialize all lower priority strategies, redefine n to be large, declare that the strategy has seen convergence for each $i' < i$ for which $\Gamma_{i'}(\zeta_{i'}; m) \downarrow$, and perform no further action at this stage. If not then proceed to (2).
- (2) If there exists n' such that $n < n'$ and $n' \in W_{i,s} - W_{i,s-1}$, then enumerate n' into B , enumerate m into C , declare the strategy to be satisfied and initialize all lower priority strategies.

The verification. It is clear that each strategy is initialized a finite number of times, and since each marker for a strategy \mathcal{P}_i is only redefined a finite number of times it follows that each \mathcal{P}_i requirement is satisfied. That the complement of B is infinite follows since markers are chosen to be large. In order to show that the \mathcal{Q}_i requirement is satisfied, let s_0 be such that the \mathcal{Q}_i strategy is never initialized after stage s_0 . It suffices to show that if D_i is total and the length of agreement is unbounded, then D_i is not compatible with any ζ for which we enumerate an axiom subsequent to stage s_0 , of the form $\Gamma_i(\zeta; n) = 0$ for some n which is enumerated into B . This suffices, because then the instructions for \mathcal{Q}_i clearly ensure that $\Gamma_i(D_i)$ is total.

So suppose that n is enumerated into B by a strategy $\mathcal{P}_{i'}$ for $i' > i$ at a stage $s_1 > s_0$ and that, prior to this enumeration, we have enumerated an axiom $\Gamma_i(\zeta; n) = 0$. Let i'' be the least such that $\mathcal{P}_{i''}$ enumerates a number into B at a stage $s_2 \geq s_1$, and such that $\mathcal{P}_{i''}$ is not initialized at any stage in the interval $[s_1, s_2]$ (so that $i'' \leq i'$). Then $\mathcal{P}_{i''}$ has already seen convergence for i when the axiom $\Gamma_i(\zeta; n) = 0$ is enumerated, since its agitator m is less than or equal to that of $\mathcal{P}_{i'}$. Let ϕ and ζ' be, respectively, the initial segments of B and ζ_i used in the computation $\Psi_i(B \oplus \zeta_i; m)$ when $\mathcal{P}_{i''}$ saw convergence for i . Since $\mathcal{P}_{i''}$ initialized all lower priority strategies when it saw convergence for i , $\zeta' \subseteq \zeta$, and ϕ is an initial segment of the final value B . It follows that if the length of agreement is unbounded then ζ cannot be an initial segment of D_i , because $\Psi_i(\phi \oplus \zeta'; m) = 0$

and $\mathcal{P}_{i''}$ enumerates m into C .

Proving the full theorem. In order to prove the full theorem we consider a version of the construction just described. Now, however, we suppose given an approximation $\{A_s\}_{s \in \omega}$ to A which is of low fixed point free degree, we consider $\Psi_i(A)$ rather than $\Psi_i(B \oplus C)$, and we must ensure that $B \oplus C$ is computable in A . The requirements \mathcal{P}_i are just as before. The new \mathcal{Q}_i requirements are:

\mathcal{Q}_i : If $\Psi_i(A)$ ($= D_i$ say) is total and $\Psi_i(B \oplus D_i) = C$ then $\Gamma_i(D_i) = B$;

So we still construct B and C and these sets will play the same roles as before – B will be the set whose degree doesn't have a joining partner, and C provides the agitators which are used in order to satisfy the requirements \mathcal{P}_i . Since A is of fixed point free degree there exists Φ such that $f = \Phi(A)$ is d.n.c. – we therefore allow our approximation to A to specify an approximation $\{f_s\}_{s \in \omega}$ to f . We use Kučera's technique of fixed point free permitting to ensure that B and C are computable in A .

The potential problem now, is that when \mathcal{P}_i sees convergence for $i' < i$ because $\Gamma_{i'}(\zeta; m) \downarrow$ for some $\zeta \subseteq \zeta_{i'}$, it will initialize all lower priority strategies as before, but now $\zeta_{i'}$ may become incompatible with ζ at subsequent stages. If at such a subsequent stage it sees convergence for i' again, then it must once again initialize all lower priority strategies, because the initial segment of B that it needs to preserve may now be longer. This might happen an infinite number of times, causing \mathcal{P}_i to initialize lower priority strategies an infinite number of times. In order to avoid this problem, we use the fact that A is of low degree. The construction specifies Turing functionals which compute B and C from A . By the recursion theorem we may assume given indices for these reductions in advance, and thus, for any m and i' , we can approximate during the construction whether or not there exist ψ, ϕ which are initial segments of (the final values) A and B respectively, such that if $\zeta = \Psi_{i'}(\psi)$ then $C \upharpoonright (m+1) \subseteq \Psi_{i'}(\phi \oplus \zeta)$. We only define $\Gamma_{i'}$ on argument m when our approximation tells us that such initial segments exist. This, then, ensures that \mathcal{P}_i only initializes strategies of lower priority finitely many times.

The terminology we use is the same as that used in the section on d.n.c. functions in Nies' book [AN], which gives a very clear introduction to fixed point free permitting.

The details are as follows. During the construction we build a partial computable function v . By the recursion theorem we may assume given an index for v and so a computable function g such that for all i , $v(i) \simeq \Psi_{g(i)}(\emptyset; g(i))$. By the recursion theorem we may also assume given a computable function h such that, for all m and i , $\lim_s h(i, m, s)$ is equal to 1 if there exist ψ, ϕ which are initial segments of A and B respectively, such that if $\zeta = \Psi_i(\psi)$ then $C \upharpoonright (m+1) \subseteq \Psi_i(\phi \oplus \zeta)$, and is equal to 0 otherwise. At any point during the construction we let ζ_i denote the finite binary string $\Psi_i(A)$. As before, the length of agreement for \mathcal{Q}_i at any point during the construction is the greatest m such that $C \upharpoonright m \subseteq \Psi_i(B \oplus \zeta_i)$. If $\Gamma_i(\zeta_i; m) \downarrow$ at any stage, then the *initial segment of ζ_i used on argument m* is the shortest $\zeta \subseteq \zeta_i$ such that $\Gamma_i(\zeta; m) \downarrow$. The modified strategies work as follows.

The \mathcal{Q}_i strategy. At each stage at which it is run the strategy considers the least n such that $\Gamma_i(\zeta_i; n) \uparrow$ and if n is less than the length of agreement and $h(i, n, s) = 1$

then it proceeds as follows. Let ϕ and ζ be, respectively, the initial segments of B and ζ_i used in the computation $\Psi_i(B \oplus \zeta_i; n) = C(n)$. If there exists a shortest ζ' with $\zeta \subseteq \zeta' \subseteq \zeta_i$ such that $\Gamma_i(\zeta'; n') \downarrow$ for all $n' < n$ and for which it is consistent with all axioms previously enumerated to enumerate the axiom $\Gamma_i(\zeta'; n) = \phi(n)$, then enumerate this axiom.

The \mathcal{P}_i strategy. Once it has been declared satisfied, the strategy performs no further actions even if subsequently initialized, since then the requirement has been permanently satisfied. When the strategy is first run (subsequent to any initialization) it chooses a large agitator m and then a large marker $n > m$. At every subsequent stage s at which it has not yet been declared satisfied it performs the following steps:

- (1) Check to see whether, for all t with $m \leq t \leq s$, $f_t(g(i)) = f_s(g(i))$. If not then initialize this strategy together with all strategies of lower priority, performing no more actions at this stage. Otherwise proceed to (2).
- (2) Check to see whether there exists any $i' < i$ for which the strategy has not previously seen convergence for any (i', ζ) such that ζ is an initial segment of the present value $\zeta_{i'}$, and such that $\Gamma_{i'}(\zeta_{i'}; m) \downarrow$. If so then perform steps (a) to (c) below, performing no further action at this stage. If not then proceed to (3).
 - (a) Initialize all lower priority strategies;
 - (b) Redefine n to be large;
 - (c) For each $i' < i$ such that $\Gamma_{i'}(\zeta_{i'}; m) \downarrow$, let ζ be the initial segment of $\zeta_{i'}$ used on argument m , and declare that the strategy has seen convergence for (i', ζ) .
- (3) If there exists n' such that $n < n'$ and $n' \in W_{i,s} - W_{i,s-1}$, then define $v(i) = f_s(g(i))$, enumerate n' into B , enumerate m into C , declare the strategy to be satisfied and initialize all lower priority strategies.

The verification. First we show that $B \oplus C$ is computable in A . Since C computes B by simple permitting, it suffices to show that C is computable in A . Given input m , use f to compute $t > m$ such that for all $i \leq m$, $f_t(g(i)) = f(g(i))$. Then m is in C only if it has been enumerated in by stage t , since if we enumerate m into C at a stage $s > t$ then this is done by a strategy \mathcal{P}_i for $i < m$ and this strategy would have been initialized (and so be forced to choose an agitator greater than m) unless for all t' with $m \leq t' \leq s$, $f_{t'}(g(i)) = f_s(g(i))$. Thus the value $v(i) = \Psi_{g(i)}(\emptyset; g(i)) = f_s(g(i))$ we define at stage s equals $f(g(i))$ and therefore f is not a d.n.c. function, which gives a contradiction.

Next we must show that, for any n, i , there are only a finite number of ζ for which we enumerate an axiom $\Gamma_i(\zeta; n) = j$ (for any $j \in \{0, 1\}$). If $\lim_s h(i, n, s) = 0$ then this is clear, so suppose $\lim_s h(i, n, s) = 1$. Let ζ and ϕ be the shortest initial segments of the final values of D_i and B such that $\Psi_i(\phi \oplus \zeta) \supseteq \tau$, where τ is $C \upharpoonright (n+1)$. If $C(n) = 0$ then the result is clear because then it will always be consistent to enumerate the axiom $\Gamma_i(\zeta; n) = 0$, so suppose otherwise. Let m be the length of the longest string ζ_0 for which we enumerate an axiom $\Gamma_i(\zeta_0; n) = 0$, and let $m' = \max\{|\zeta|, |\zeta_0|\}$. Then at stages s such that $\phi \subset B_s$ we do not enumerate any axiom $\Gamma_i(\zeta_1; n) = 1$ for $\zeta_1 \supseteq \zeta$ of length greater than m' .

It follows that each strategy is initialized only finitely often and that the marker for each \mathcal{P}_i requirement is redefined only a finite number of times. Therefore each

\mathcal{P}_i requirement is satisfied. That the complement of A is infinite follows since markers are chosen to be large. That the requirement \mathcal{Q}_i is satisfied follows from almost exactly the same argument as before. Let s_0 be such that the \mathcal{Q}_i strategy is never initialized after stage s_0 . Suppose that n is enumerated into B by a strategy $\mathcal{P}_{i'}$ for $i' > i$ at a stage $s_1 > s_0$ and that, prior to this enumeration, we have enumerated an axiom $\Gamma_i(\zeta; n) = 0$. When this axiom is enumerated, $\mathcal{P}_{i'}$ has seen convergence for some (i, ζ_0) such that $\zeta_0 \subseteq \zeta$. Let i'' be the least such that $\mathcal{P}_{i''}$ enumerates a number into B at a stage $s_2 \geq s_1$, and such that $\mathcal{P}_{i''}$ is not initialized at any stage in the interval $[s_1, s_2]$ (so that $i'' \leq i'$). Then $\mathcal{P}_{i''}$ has already seen convergence for some (i, ζ_1) such that $\zeta_1 \subseteq \zeta_0$ when the axiom $\Gamma_i(\zeta; n) = 0$ is enumerated, since its agitator m is less than or equal to that of $\mathcal{P}_{i'}$. Let ϕ be the initial segment of B used in the computation $\Psi_i(B \oplus \zeta_i; m)$ when $\mathcal{P}_{i''}$ saw convergence for (i, ζ_1) . Since $\mathcal{P}_{i''}$ initialized all lower priority strategies when it saw convergence for (i, ζ_1) , ϕ is an initial segment of the final value B . It follows that if the length of agreement is unbounded then ζ cannot be an initial segment of D_i , because $\Psi_i(\phi \oplus \zeta_1; m) = 0$ and $\mathcal{P}_{i''}$ enumerates m into C .

Final remarks. The proof given here actually suffices to show that every low c.e. degree is bounded by a low c.e. degree which does not satisfy join. Given W of low c.e. degree, let A be of low fixed point free degree above that of W . The construction ensures that $W \oplus B \oplus C$ does not satisfy the join property.

REFERENCES

- [BC] S.B. Cooper, The strong anticupping property for recursively enumerable degrees, *Journal of Symbolic Logic*, vol. 54, 527-539, 1989.
- [DJS] R. Downey, C.G. Jockusch, M. Stob, Array non-recursive degrees and genericity, *Computability, Enumerability, Unsolvability*, vol 224 of the London Mathematical Society Lecture Note Series, 93-104, 1996.
- [N4] R. Downey, N. Greenberg, A.E.M. Lewis, A. Montalbán, Extensions of uppersemilattice embeddings below computably enumerable degrees, to appear.
- [GMS] N. Greenberg, A. Montalbán, R. Shore, Generalized high degrees have the complementation property, *Journal of Symbolic Logic*, 69, 1200-1220, 2004.
- [AK] A. Kučera, Measure, Π_1^0 -classes and complete extensions of PA, *Recursion Theory Week* (Oberwolfach, 1984), volume 1141 of Lecture Notes in Math. 245-259, Springer, Berlin, 1985.
- [ML] M. Lerman, Degrees of Unsolvability, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.
- [AN] A. Nies, Computability and Randomness, Oxford Logic Guides 51, Oxford University Press, 2009.
- [PR] D.B. Posner, R.W. Robinson, Degrees joining to $\mathbf{0}'$, *Journal of Symbolic Logic*, vol 46, no. 4, 714-722, 1981.
- [RR] R.W. Robinson, Degrees joining to $\mathbf{0}'$, *Notices of the American Mathematical Society*, vol 19, 1972.
- [SS] T.A. Slaman, J.R. Steel, Complementation in the Turing degrees, *Journal of Symbolic Logic*, vol 54, no. 1, 160-176, 1989.
- [RS] R.I. Soare, *Recursively enumerable sets and degrees*, Springer, New York, (1987).

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