PROPERTIES OF THE JUMP CLASSES

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1. Introduction

On the Turing degree structure we consider an ordering relation, which is that inherited by the Turing reducibility on sets of natural numbers, and also the jump function, which arises naturally as the function on degrees induced by the operator which takes each set to its halting problem. In order to understand the structure it seems a basic task to understand the relationship between these two objects. The aim of this paper is to advocate a line of research which looks to better understand this relationship by establishing the order theoretic properties of degrees in each of the various jump classes. We shall survey a range of results in the area and detail a number of open questions. Some of these questions have been noted in the existing literature, while many seem not to have been previously addressed and may well be amenable to attack by known techniques.

For the degrees below $\mathbf{0}'$, the jump hierarchy was originally introduced by Cooper in [BC] and Soare in [RS].

Definition 1.1. A degree $\boldsymbol{a} \leq \mathbf{0}'$ is *low*$_n$ if $\boldsymbol{a}^{(n)} = \mathbf{0}^{(n)}$, where $\boldsymbol{a}^{(n)}$ is the nth jump of $\boldsymbol{a}$. A degree $\boldsymbol{a} \leq \mathbf{0}'$ is *high*$_n$ if $\boldsymbol{a}^{(n)} = \mathbf{0}^{(n+1)}$.

Since the jump function is order preserving, being low$_n$, or being high$_n$, may be seen to formalize being close to $\mathbf{0}$ on the one hand, or close to $\mathbf{0}'$ on the other. As originally suggested by Jockusch and Posner [JP], these notions can then be appropriately adapted to deal with degrees which are not necessarily below $\mathbf{0}'$.

Definition 1.2. For $n \geq 1$, a degree $\boldsymbol{a}$ is *generalized low*$_n$ (GL$_n$) if $\boldsymbol{a}^{(n)} = (\boldsymbol{a} \vee \mathbf{0}')^{(n-1)}$. A degree $\boldsymbol{a}$ is *generalized high*$_n$ (GH$_n$) if $\boldsymbol{a}^{(n)} = (\boldsymbol{a} \vee \mathbf{0}')^{(n)}$.

Note that these two hierarchies coincide below $\mathbf{0}'$. A notable feature of the generalized jump classes, however, is that they do not respect the ordering relation on the Turing degrees in the way that one might initially expect. Every generalized high (GH$_1$) that is not above $\mathbf{0}'$, for example, is bounded by a generalized low (GL$_1$) – this is easily seen by relativizing, to any such generalized high, the proof that every non-zero degree below $\mathbf{0}'$ can be joined to $\mathbf{0}'$ by a low degree. In most cases, however, the definition given does seem to be the correct one for dealing with degrees outside the local structures; results that hold for the high degrees, or high$_2$ degrees, or degrees which are not low$_2$, for example, can very often be modified in order to work for the respective generalized jump classes. For our purposes here, the order theoretic quirk of the generalized jump classes noted above, is probably most fruitfully seen as a feature of the Turing degrees rather than an inadequacy

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in the definition.

In what follows, we shall consider various order theoretic properties, such as the *cupping* property, the *join* property, and so on (all of which will subsequently be defined), and we shall look to establish for which jump classes it is the case that membership either ensures satisfaction or ensures failure of the property. One motivation here is quite simply that there are many basic open questions along these lines, the answers to which will tell us a lot about the Turing degree structure. Further motivation is provided by our desire for natural definability results. For all jump classes other than low, we know thanks to Nies, Shore and Slaman [NSS], that these jump classes are definable. With these results established through the use of coding techniques, however, so far there is a conspicuous lack of *natural* definability results for the jump classes. Roughly speaking, by a natural definition for a jump class, we mean a simple order theoretic property (i.e. not too long and without too many alternations of quantifiers) which is satisfied by a degree iff it belongs to that jump class. Results of this kind have long been sought by researchers in the area. The basic idea, then, is that we shall start by considering very simple order theoretic properties, and then work our way up to considering properties which are more complex – the hope is that at some point during this process, definability results will precipitate out.

Notation and terminology will be standard unless explicitly stated otherwise, and will generally follow [BC2] and [RS2].

2. The cupping property

We begin by considering a very simple property, for which the relationship with the jump classes is already well understood.

Definition 2.1. A degree $\mathbf{a}$ satisfies the cupping property if, for all $\mathbf{b} \geq \mathbf{a}$, there exists $\mathbf{c} < \mathbf{b}$ with $\mathbf{a} \vee \mathbf{c} = \mathbf{b}$.

Theorem 2.2 (Jockusch,Posner [JP])**.** *All non-GL_2 degrees satisfy the cupping property.*

On the other hand, initial segment results show that there are degrees which are low, and also degrees which are low_2 non-low, which do not satisfy the cupping property (see [ML] for a thorough survey of initial segment results). Recall that a degree is PA if it contains a set which effectively codes a complete and consistent extension of Peano Arithmetic.

Theorem 2.3 (Kučera [AK])**.** *All PA degrees satisfy the cupping property.*

For another interesting proof of this theorem, see [AL]. By the low basis theorem [JS], there are PA degrees which are low, and since the PA degrees are upward closed, it follows that there are PA degrees which are low_2 non-low. To summarize, then, while all non-GL_2 degrees satisfy the cupping property, being low or being low_2 non-low, does not suffice to ensure either satisfaction or failure of the property.

Perhaps it is interesting at this point to comment on the techniques which are used in order to prove Theorem 2.2. As is almost almost always going to be the case when proving that all non-GL_2 degrees satisfy a certain property, it is domination techniques which lie at the heart of transferring results which are already known to be true for the degrees above $\mathbf{0}'$. Given a proof for the degrees above $\mathbf{0}'$ one first observes that, in fact, it is the ability of $\emptyset'$ to compute some sufficiently fast growing function f which allows the proof to go through. This function f may be the length of time needed to search for splittings above a given set of strings until a splitting is found above each string where splittings exist, for example. Now if a degree $\boldsymbol{a}$ below $\mathbf{0}'$ is non-low$_2$ then $\mathbf{0}'$ is not high relative to it, and $\boldsymbol{a}$ therefore computes a function g which is not dominated by f. Showing that the result can be transferred to $\boldsymbol{a}$ then amounts to showing that, with appropriate modifications, having infinitely many arguments n for which $g(n) > f(n)$ suffices for us to define an $\boldsymbol{a}$ computable construction in which g now plays the role that f played in the $\emptyset'$ oracle construction, and which now gives the result for $\boldsymbol{a}$. When $\boldsymbol{a}$ is non-GL$_2$ it is then $\boldsymbol{a} \vee \mathbf{0}'$ which must take the place of $\mathbf{0}'$.

While these observations complete the picture as far as interactions with the jump classes are concerned, we detail below some basic open questions concerning the cupping property which remain.

Definition 2.4. A degree $\boldsymbol{b}$ is a *strong minimal cover* for $\boldsymbol{a}$ if the degrees strictly below $\boldsymbol{b}$ are precisely those below and including $\boldsymbol{a}$.

Question 2.5. Does every degree either satisfy the cupping property or have a strong minimal cover?

There is another interesting question, which concerns characterizing those degrees which satisfy the cupping property.

Definition 2.6. For $i \in \{0,1\}$ let $\bar{i} = 1 - i$. We say $T : 2^{<\omega} \to 2^{<\omega}$ is a *tree* if, whenever $T(\sigma * i)\downarrow$ for $i \in \{0,1\}$:

(i) $T(\sigma)\downarrow \subset T(\sigma * \bar{i})\downarrow$ and;
(ii) $T(\sigma * i)$ and $T(\sigma * \bar{i})$ are incompatible.

Following the standard abuse of notation, we say $\sigma \in T$ when σ is in the range of T. A is a path through T if infinitely many initial segments of A are in T. We say a tree T is *perfect* if it is a total function, and is $\mathbf{a}$-*incapable* if no path through T is of degree $\geq \mathbf{a}$.

It is not difficult to see that if there is a perfect $\mathbf{a}$-incapable tree T of degree $\leq \boldsymbol{a}$, then $\boldsymbol{a}$ satisfies the cupping property. Let A be of degree $\boldsymbol{a}$, and let B be of degree above $\boldsymbol{a}$. Then consider $C \leq_T B$ which is $\bigcup_i T(B \upharpoonright i)$. Since T is $\boldsymbol{a}$-incapable it follows that $C <_T B$. Given T and C we can compute B, so that $B \equiv_T A \oplus C$.

Recall that a degree is a.n.r. (array non-recursive) if, for every $f <_{wtt} K$, there exists g of degree $\boldsymbol{a}$ which is not dominated by f. Note that all non-GL$_2$ degrees are a.n.r. and in [DJS] it is observed that many properties that hold of all non-GL$_2$ degrees can be shown to hold also for the a.n.r. degrees. In particular, they show that the a.n.r. degrees satisfy the cupping property and in [AL2] it is shown that if $\boldsymbol{a}$ is a.n.r. then it computes a perfect $\mathbf{a}$-incapable tree. As noted above, Kučera showed that all PA degrees satisfy the cupping property, and in [AL] it is shown that if $\boldsymbol{a}$ is PA then it computes a perfect $\mathbf{a}$-incapable tree. The following question therefore seems reasonable:

Question 2.7. Are the degrees which satisfy the cupping property precisely those degrees $\mathbf{a}$ which compute perfect $\mathbf{a}$-incapable trees?

A positive answer to this question would suffice to show that no minimal degree satisfies the cupping property. This follows since every perfect tree of minimal degree has a path which computes the tree [AL3].

3. The join property

We turn now to a property, which is richer in this context, essentially because it is not upward closed.

Definition 3.1. A degree $\mathbf{a}$ satisfies the join property if, for all non-zero $\mathbf{b} < \mathbf{a}$ there exists $\mathbf{c} < \mathbf{a}$ with $\mathbf{b} \vee \mathbf{c} = \mathbf{a}$.

The strongest result here is as follows:

Theorem 3.2 (Downey, Greenberg, Lewis, Montalbán, [DGLM])**.** *All degrees which are non-GL_2 satisfy the join property.*

Initial segment results can be used in order to show that all possibilities can be realized for the low and for the low_2 non-low degrees, i.e. there exist low degrees which satisfy the join property and low degrees that don't, and similarly for the degrees which are low_2 non-low.

The situation becomes more interesting, however, when we consider the upward closure – for which degrees is it the case that all degrees above satisfy the join property? In looking to answer this question, it becomes natural to consider upward closed classes of degrees which we are used to working with. Do all PA degrees satisfy the join property? Do all a.n.r. degrees satisfy the join property? Another natural question is as to whether satisfaction of the cupping property is equivalent to all degrees above satisfying join. We can answer all of these questions with the following result:

Definition 3.3. A function $f : \omega \rightarrow \omega$ is fixed-point-free if, for all n, $\phi_n \neq \phi_{f(n)}$, where ϕ_n is the nth partial computable function in some fixed effective listing of all such functions. A degree is fixed-point-free if it contains a function which is fixed-point-free.

Theorem 3.4 (Lewis [AL4])**.** *All low fixed-point-free degrees fail to satisfy the join property.*

The techniques used to prove this theorem are a combination of Kučera's fixed-point-free permitting below $\emptyset'$ and the techniques developed by Slaman and Steel [SS] in order to show that there exist c.e. degrees which do not satisfy the join property (when considered as an element of the Δ^0_2 degrees).

Corollary 3.5. Above each low degree, there is a low degree which doesn't satisfy join.

Proof. Each low degree is bounded by a degree which is low and fixed-point-free. □

Corollary 3.6. There are PA/a.n.r./Martin-Löf random degrees which don't satisfy join.

Proof. All PA degrees and all Martin-Löf random degrees are fixed-point-free, and it follows from the low basis theorem that there exist PA degrees which are low and also Martin-Löf random degrees are low. For the a.n.r. degrees, the corollary follows from Corollary 3.5 and the fact that the a.n.r. degrees are upward closed. □

For an introduction to the study of algorithmic randomness (and, in particular, Martin-Löf randomness), we refer the reader to [AN] and [DH].

Corollary 3.7. Satisfaction of the cupping property is not equivalent to all degrees above satisfying join.

Proof. Any PA degree properly bounds another PA degree. The result then follows from Corollary 3.6 and the fact that all PA degrees satisfy the cupping property. □

There is another theorem along the same lines:

Theorem 3.8 (Lewis [AL4]). *Above every low c.e. degree, there is a low c.e. degree which doesn't satisfy join.*

This leaves open various questions. Working below $\mathbf{0}'$ first of all, we have already managed to separate the non-low$_2$ degrees from the low degrees, and it seems natural to ask if this can be extended to give a definition of low$_2$:

Question 3.9. In $\mathcal{D}[\leq \mathbf{0}']$, are the low$_2$ degrees precisely those for which it is not the case that all degrees above satisfy join?

It remains open, in fact, whether $\mathbf{0}'$ can be given an extremely simple definition using the join property, although one would expect the following question to be answered in the negative:

Question 3.10. Can $\mathbf{0}'$ be defined as the least degree such that all degrees above satisfy join?

There is an interesting observation to be had here. Let us suppose for a moment that question 3.10 receives a negative response, and that, in fact, any degree which bounds a high degree satisfies join. This moves us towards a definition of $\mathbf{0}'$ in another direction. Immediately from this we get a formula sufficient to distinguish $\mathbf{0}'$ from all degrees comparable with it, because then all degrees above $\mathbf{0}'$ satisfy the following formula, while no degree strictly below $\mathbf{0}'$ does so:

$$\mathbf{a} \text{ satisfies mincup, and } \forall \mathbf{b} \geq \mathbf{a},\ \forall \text{ minimal degrees } \mathbf{m} < \mathbf{b}, \mathbf{b} \text{ satisfies join}(\mathbf{m}),$$

where $\mathbf{a}$ satisfies mincup if for all $\mathbf{b} \geq \mathbf{a}$ there exists a minimal degree $\mathbf{m} < \mathbf{b}$ with $\mathbf{a} \vee \mathbf{m} = \mathbf{b}$, and where join($\mathbf{m}$) means join above $\mathbf{m}$, i.e. $\mathbf{b}$ satisfies join($\boldsymbol{m}$) if, for all $\boldsymbol{c}$ with $\boldsymbol{m} < \boldsymbol{c} < \boldsymbol{b}$ there exists $\mathbf{d}$ with $\boldsymbol{m} < \boldsymbol{d} < \boldsymbol{b}$ and $\mathbf{d} \vee \mathbf{c} = \mathbf{b}$. Let us see first why degrees above $\mathbf{0}'$ satisfy the formula above. All such degrees satisfy mincup – this follows from the fact that $\emptyset'$ computes a perfect tree of sets of minimal degree. If $\mathbf{b} \geq \mathbf{0}'$ bounds a minimal degree $\mathbf{m}$ then, since all minimal degrees are GL$_2$, $\mathbf{b}$ bounds a degree which is high relative to $\mathbf{m}$ and so satisfies join($\mathbf{m}$) according to our hypothesis (given relativization).

Now suppose that $\mathbf{a} < \mathbf{0}'$. Then $\mathbf{a}$ may not satisfy mincup, but if it does then let $\mathbf{b} \geq \mathbf{a}$ be GL$_1$ (such a degree will exist, since any degree which is not above $\mathbf{0}'$ is bounded by a degree which is generalized low), and let $\mathbf{m}$ be a minimal degree such that $\mathbf{m} \vee \mathbf{a} = \mathbf{b}$. Then $\mathbf{m} \vee \mathbf{0}' \geq \mathbf{b} \vee \mathbf{0}' = \mathbf{b}'$, so $\mathbf{m} \vee \mathbf{0}' = \mathbf{b} \vee \mathbf{0}' = \mathbf{b}' = \mathbf{m}'$, so that $\mathbf{b}$ is actually low over $\mathbf{m}$. By Corollary 3.5, above every low degree there exists a low degree which doesn't satisfy join. This proof relativizes, meaning that we may choose $\mathbf{c} \geq \mathbf{b}$ which doesn't satisfy join($\mathbf{m}$).

One may reasonably ask how useful this observation actually is, given that we are supposing something that may well not be true – it could well be the case that there are degrees which bound a high degree and which do not satisfy the join property. The point here, though, is that we don't have to consider just the join property. Any property which suffices to separate high and low cones will suffice:

Question 3.11. Can we find a natural order theoretic property P, which suffices to separate the high and low cones, in the sense that above any low degree there is a degree which does not satisfy P, while any degree which bounds a high degree satisfies P?

Question 3.12. If $\mathbf{a} \not\geq \mathbf{0}'$ does there necessarily exist $\mathbf{b} \geq \mathbf{a}$ and a minimal degree $\mathbf{m} < \mathbf{b}$ such that $\mathbf{b}$ is low over $\mathbf{m}$?

These two questions are interesting because positive answers to both would give a natural definition of the jump (presuming necessary relativizations hold). This follows because then $\mathbf{0}'$ would be the least degree such that:

$$\forall \boldsymbol{b} \geq \boldsymbol{a}, \forall \text{ minimal degrees } \boldsymbol{m} < \boldsymbol{b},\ \boldsymbol{b} \in P(\mathbf{m}),$$

where $P(\mathbf{m})$ denotes P above $\mathbf{m}$, in just the same way that we let join($\mathbf{m}$) denote join above $\mathbf{m}$ previously.

4. The join of minimals property

Another basic question is as to which degrees satisfy the property of bounding a minimal degree. The strongest result here is due to Jockusch, who made a new use of the recursion theorem in the context of working with high and generalized high degrees, in order to extend the result of Cooper that every high degree bounds a minimal degree:

Theorem 4.1 (Jockusch [CJ]). *Every generalized high degree bounds a minimal degree.*

The fact that this is sharp, follows from a difficult result of Lerman's:

Theorem 4.2 (Lerman [ML2]). *There exists a high$_2$ degree which does not bound any minimal degree.*

We consider next those degrees which are the join of two minimal degrees. The strongest result here is:

Theorem 4.3 (Ellison, Lewis [EL]). *All GH_1 degrees are the join of two minimal degrees.*

This settles a conjecture of Posner's from the 70s. The previous results were as follows. First Cooper showed that $\mathbf{0}'$ is the join of two minimal degrees [BC3], and then Posner [DP] extended this to all degrees above $\mathbf{0}'$. It follows from Theorem 4.2 that Theorem 4.3 is sharp in terms of the jump hierarchy.

Posner asked, in fact, whether $\mathbf{0}'$ could be defined as the least degree such that all degrees above are the join of two minimal degrees. This was answered by Shore [SH] who showed that there exists a degree incomparable with $\mathbf{0}'$ such that all degrees above are the join of two minimal degrees. This immediately gives members of each generalized jump class which are the join of two minimal degrees. The following question remains open:

Question 4.4. Is $\mathbf{0}'$ minimal amongst the degrees such that all degrees above are the join of two minimal degrees?

5. The minimal cupping property

We gave the definition earlier:

Definition 5.1. A degree $\mathbf{a}$ satisfies the minimal cupping property (mincup) if for all $\mathbf{b} \geq \mathbf{a}$ there exists a minimal degree $\mathbf{m} < \mathbf{b}$ with $\mathbf{a} \vee \mathbf{m} = \mathbf{b}$.

Let us consider the situation below $\mathbf{0}'$ first.

Proposition 5.2. In $\mathcal{D}[\leq \mathbf{0}']$, all high degrees satisfy the minimal cupping property.

Proof. We give only a sketch of the proof, for those familiar with minimal degree constructions.

Suppose given $A \leq_T B$ which are both of high degree. We must construct $C \leq_T B$ such that $A \oplus C \equiv_T B$ and C is of minimal degree. To do so, apply Jockusch's use of the recursion theorem for building a minimal degree below a high degree [CJ] in order to construct C which is computable in B, but this time, since A is high, suppose that the function which approximates whether every initial segment of C has Ψ-splittings above it in any given splitting tree, is computable in A. At the end of each stage $s+1$ we code $B(s)$ with our choice as to whether our next initial segment of C should be the leftmost or the rightmost string in the splitting found at that stage. $A \oplus C$ can retrace the construction and so compute B.

□

This result is clearly sharp in terms of the jump hierarchy – again this follows from Lerman's result [ML2] that there exist high$_2$ degrees which don't bound minimal degrees.

Conjecture 5.3 (Lewis)**.** [1] *In $\mathcal{D}[\leq \mathbf{0}']$, a c.e. degree is high iff it satisfies the minimal cupping property.*

If this conjecture holds then it this gives us a nice characterization of the high c.e. degrees (and gives a natural definition of the high c.e. degrees in terms of a definition for the c.e. degrees in $\mathcal{D}[\leq \mathbf{0}']$). Unfortunately it certainly can't be extended to give us a natural definition of the high degrees:

Theorem 5.4 (Ellison, Lewis [EL2])**.** *There exists a low degree which satisfies the minimal cupping property.*

Note that this theorem holds globally. In order to give this result, the approach taken is to establish the existence of a perfect tree which is of low degree, and such that every path through the tree is of minimal degree.

The first remaining questions here, also concern the global structure.

Question 5.5. Do all GH_1 degrees satisfy mincup?

[1] It should be noted that in a previous version of this paper, this was claimed as a result when it should not have been.

6. The meet and complementation properties

We consider next, the meet and complementation properties:

Definition 6.1. A degree $\boldsymbol{a}$ satisfies the meet property if, for all $\boldsymbol{b} < \boldsymbol{a}$, there exists non-zero $\boldsymbol{c} < \boldsymbol{a}$ such that $\boldsymbol{b} \wedge \boldsymbol{c} = \mathbf{0}$.

Definition 6.2. A degree $\boldsymbol{a}$ satisfies the complementation property if, for all non-zero $\boldsymbol{b} < \boldsymbol{a}$, there exists $\boldsymbol{c} < \boldsymbol{a}$ such that $\boldsymbol{b} \wedge \boldsymbol{c} = \mathbf{0}$ and $\boldsymbol{b} \vee \boldsymbol{c} = \boldsymbol{a}$.

First of all, let us establish the uninteresting cases – initial segment results can be used in order to show that all possibilities can be realized for the low and for the low_2 non-low degrees. On the positive side, Posner [DP2] showed that $\mathbf{0}'$ satisfies the complementation property using a non-uniform proof. This non-uniformity was subsequently shown not to be necessary by Slaman and Steel [SS]. The following is the strongest result known:

Theorem 6.3 (Greenberg, Montalbán and Shore [GMS])**.** *All generalized high degrees satisfy the complementation property.*

The proof given for this stronger result is once again non-uniform, and it remains open as to whether the complement here can be constructed uniformly. For all other jump classes, interaction with these two properties remains open:

Question 6.4. Do all non-GL_2 satisfy complementation/meet? All GH_2?

We would conjecture that there exists a high_2 degree which doesn't satisfy the meet property, so that these questions all receive a negative answer. At the time of writing, we are examining a potential proof of the following result:

Conjecture 6.5 (Lewis, Ng)**.** *All c.e. degrees satisfy the meet property i.e. if $\boldsymbol{a}$ is c.e. then for* all $\boldsymbol{b} < \boldsymbol{a}$*, there exists non-zero $\boldsymbol{c} < \boldsymbol{a}$ (which is not necessarily c.e.) such that $\boldsymbol{b} \wedge \boldsymbol{c} = \mathbf{0}$.*

A positive solution to this question would settle a conjecture of Cooper's from the 80s. The following question was asked by Slaman and Steel:

Question 6.6 (Slaman,Steel [SS])**.** Can $\mathbf{0}'$ be defined as the least degree such that all degrees above satisfy complementation?

As with Question 3.10, the expectation is presumably that this question will eventually be answered in the negative, but that the counter example may be difficult to construct.

7. The capping property

In this section, we briefly consider a property about which little is known:

Definition 7.1. A degree $\boldsymbol{a}$ satisfies the capping property if, for all $\boldsymbol{b} > \boldsymbol{a}$ there exists non-zero $\boldsymbol{c} < \boldsymbol{b}$ such that $\boldsymbol{a} \wedge \boldsymbol{c} = \mathbf{0}$.

It follows from Theorem 5.4, that there exists a low degree such that all degrees above satisfy the capping property. This suffices to show that there exist members of each jump class which satisfy the capping property. It also follows from Proposition 5.2 that, in $\mathcal{D}[\leq \mathbf{0}']$, all high degrees satisfy the capping property. Nothing else substantial is known at this point.

Question 7.2. Do all GH_1 degrees satisfy the capping property?

8. The minimal complementation property

In this final section, we consider the most complex property so far:

Definition 8.1. A degree $\boldsymbol{a}$ satisfies the minimal complementation property if, for all non-zero $\boldsymbol{b} < \boldsymbol{a}$, there exists a minimal degree $\boldsymbol{m} < \boldsymbol{a}$ such that $\boldsymbol{b} \vee \boldsymbol{m} = \boldsymbol{a}$.

The results here are as follows:

Theorem 8.2 (Lewis, Seetapun, Slaman [AL5])**.** $\mathbf{0}'$ *satisfies the minimal complementation property.*

Theorem 8.3 (Lewis [AL6])**.** *All degrees above* $\mathbf{0}'$ *satisfy the minimal complementation property.*

In fact, one can complement quite a lot of degrees simultaneously:

Theorem 8.4 (Lewis [AL7])**.** *There exists a minimal degree below* $\mathbf{0}'$ *which complements all (non-zero, incomplete) c.e. degrees.*

While it is immediately clear that no Δ^0_2 degree can complement all other non-zero and incomplete Δ^0_2 degrees, one might hope that *some* version of this theorem might hold for the Δ^0_2 degrees in general – there might be two degrees such that each non-zero, incomplete degree is complemented by at least one of them, for example. In fact, however, this fails very strongly:

Theorem 8.5 (Lewis [AL8])**.** *For every degree* $\boldsymbol{c}$ *with* $\mathbf{0} < \boldsymbol{c} \leq \mathbf{0}'$ *and every uniformly* Δ^0_2 *sequence of degrees* $\{\boldsymbol{b}_i\}_{i\geq 0}$ *such that, for all* i, $\boldsymbol{b}_i \ngeq \boldsymbol{c}$, *there exists* $\boldsymbol{a}$ *with* $\mathbf{0} < \boldsymbol{a} < \mathbf{0}'$ *such that* $\boldsymbol{a} \vee \boldsymbol{b}_i \ngeq \boldsymbol{c}$ *for all* i.

The first remaining questions are these:

Question 8.6. Do all/any incomplete high degrees satisfy the minimal complementation property?

Question 8.7. Can $\mathbf{0}'$ be defined as the least degree such that all degrees above satisfy minimal complementation?

9. Final comments

Our aim here has been to present the case that this is an area of research in which many basic and fundamental questions remain open. While we are yet to establish any natural definability results for the jump classes, the number of simple order theoretic properties for which interactions with the jump classes are not yet well understood, would certainly seem to leave open the realistic possibility of finding results of this kind.

References

[BC] Cooper,S.B., Distinguishing the arithmetical hierarchy, preprint, Berkeley, 1972.

[BC2] Cooper,S.B., *Computability Theory*, Chapman & Hall, 2004.

[BC3] Cooper,S.B., Degrees of unsolvability complementary between recusively enumerable degrees, *Annals of Mathematical Logic*, 4 (1), 31-74, 1972.

[DGLM] Downey,R., Greenberg,N., Lewis,A.E.M., Montalbán,A., Extensions of embeddings below computably enumerable degrees, to appear.

[DH] Downey,R., Hirshfeldt,D., *Algorithmic Randomness and Complexity*, Springer-Verlag, in preparation.

[DJS] Downey,R., Jockusch,C.G. and Stob,M., Array nonrecursive degrees and genericity, *Computability, Enumerability, Unsolvability*, vol. 224, London Mathematical Society Lecture Note Series, 93-104, 1996.

[EL] Ellison,P., Lewis,A.E.M., Joining up to the generalized high degrees, to appear.

[EL2] Ellison,P., Lewis,A.E.M., The minimal cupping property, to appear.

[GMS] Greenberg,N., Montalbán,A. and Shore,R.A., Generalized high degrees have the complementation property, *Journal of Symbolic Logic*, 69, 1200-1220, 2004.

[CJ] Jockusch,C.G., Simple proofs of some theorems on high degrees of unsolvability, *Canadian Journal of Mathematics*, 29, (1977), 1072-1080.

[JP] Joskusch,C.G., and Posner,D.B., Double jumps of minimal degrees, *Journal of Symbolic Logic*, 43, 715-724, 1978.

[JS] Jockusch,C.G., and Soare,R., Π^0_1 classes and degrees of theories, *Transactions of the American Mathematical Society*, 173, pp. 33-56, 1972.

[AK] Kučera,A., Measure, Π^0_1 classes and complete extensions of PA, in *Recursion Theory Week (Proceedings Oberwolfach 1984)*, Lecture Notes in Mathematics, vol 1141, pp. 245-259, 1985.

[ML] Lerman,M., *Degrees of Unsolvability*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

[ML2] Lerman,M., Degrees which do not bound minimal degrees, *Annals of Pure and Applied Logic*, 30, 249-276, 1986.

[AL] Lewis,A.E.M., Π^0_1 classes, strong minimal covers and hyperimmune-free degrees, *Bulletin of the London Mathematical Society*, 39, 892-910, 2007.

[AL2] Lewis,A.E.M., Strong minimal covers and a question of Yates: the story so far, *Proceedings of the Logic Colloquium 2006*.

[AL3] Lewis,A.E.M., On a question of Slaman and Groszek, *Proceedings of the American Mathematical Society*, 136, 3663-3668, 2008.

[AL4] Lewis,A.E.M., A note on the join property, to appear.

[AL5] Lewis,A.E.M., Minimal complements for degrees below 0', *Journal of Symbolic Logic*, 69 (4), 937-966, 2004.

[AL6] Lewis,A.E.M., The minimal complementation property above 0', *Mathematical Logic Quarterly*, 51 (5), 470-492, 2005.

[AL7] Lewis,A.E.M., A single minimal complement for the c.e. degrees, *Transactions of the American Mathematical Society*, 359, 5817-5865, 2007.

[AL8] Lewis,A.E.M., Finite cupping sets, *Archive for Mathematical Logic*, 43, 845-858, 2004.

[AN] Nies,A., *Computability and Randomness*, Oxford University Press, 2009.

[NSS] Nies,A., Shore,R.A., and Slaman,T.A., Interpretability and definability in the recursively enumerable degrees, *Proceedings of the London Mathematical Society*, (3), vol. 77, no. 2, 241291, 1998.

[DP] Posner,D., A survey of the non-r.e. degrees $\leq \mathbf{0}'$, *London Mathematical Society Lecture Note Series* 45, Recursion Theory: its Generalisations and Applications, Proceedings of the Logic Colloquium '79, Leeds, edited by F.R. Drake and S.S.Wainer.

[DP2] Posner,D., The uppersemilattice of degrees $\leq \mathbf{0}'$ is complemented, *Journal of Symbolic Logic*, 46, 705-713, 1981.

[SH] Shore,R., Natural definability in degree structures, *Computability Theory and Its Applications: Current Trends and Open Problems*, P. Cholak, S. Lempp, M. Lerman and R. A. Shore eds., Contemporary Mathematics, AMS, Providence RI, 255-272, 2000.

[SS] Slaman,T.A. and Steel,J.R., Complementation in the Turing degrees, J. Symbolic Logic, vol. 54 no. 1, pp. 160176, 1989.

[RS] Soare,R.S., Automorphisms of the lattice of recursively enumerable sets, *Bulletin of the American Mathematical Society*, 80, 53-58, 1974.

[RS2] R.I. Soare, *Recursively enumerable sets and degrees*, Springer, New York, (1987).

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