A FIXED POINT FREE MINIMAL DEGREE

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Abstract. We show that there exists a Turing degree which is minimal and fixed point free.

1. Introduction

It is the aim of this paper to give a positive solution to an old question of Sacks:

Question 1.1 (Sacks [6], 1985)**.** Does there exist a minimal degree which is fixed point free?

In order to explain what this question means we need first to establish some basic terminology and notation. Let Φ_m be the m-th partial computable function according to some fixed effective listing of all such functions. The set of all finite strings of natural numbers is denoted $\omega^{<\omega}$, and if $\tau \in \omega^{<\omega}$ then we let $|\tau|$ denote the length of τ. We let Ψ_m denote the m-th member of some fixed effective listing of the Turing functionals Ψ such that for any $f : \omega \rightarrow \omega$ and any $x \in \omega$, either $\Psi(f;x) \uparrow$ or $\Psi(f;x) \in \{0,1\}$. We assume also that for any $\tau \in \omega^{<\omega}$ and any $x, m \in \omega$, $\Psi_m(\tau;x)$ is defined only if this computation converges in $< |\tau|$ many steps and $\Psi_m(\tau;x')$ is defined for all $x' < x$. A function f is fixed point free (FPF) if for every m, $\Phi_m \neq \Phi_{f(m)}$. A function f is DNR if for every m, $f(m) \neq \Phi_m(m)$. A degree is FPF if it contains a function which is FPF, and a degree is DNR if it contains a function which is DNR. It is well known that the FPF degrees are precisely the DNR degrees. A degree $\mathbf{a} \neq \mathbf{0}$ is minimal if the only degree strictly below $\mathbf{a}$ is $\mathbf{0}$ (we shall say $\boldsymbol{a}$ is below $\boldsymbol{b}$ when $\boldsymbol{a} \leq \boldsymbol{b}$, and when we wish to indicate that $\boldsymbol{a} < \boldsymbol{b}$ we shall say that $\boldsymbol{a}$ is *strictly* below $\boldsymbol{b}$).

In privately circulated notes written in 1993, the first author provided a positive solution to question 1.1. Unfortunately the complexity of the proof combined with the dense style of presentation meant that other researchers were unable to verify correctness and this proof has remained unpublished. This paper is the result of efforts by the second author to simplify the original proof and to give an exposition which is easier to follow. The motivation of the second author in this task was provided, in part, by interest in an old question of Yates:

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Question 1.2 (Yates)**.** We say that a degree $\boldsymbol{b}$ is a strong minimal cover for $\boldsymbol{a}$ if the degrees strictly below $\boldsymbol{b}$ are precisely the degrees below $\boldsymbol{a}$. Does every minimal degree have a strong minimal cover?

Recall that a degree $\boldsymbol{a}$ is hyperimmune-free iff for every function f of degree below $\mathbf{a}$ there exists a computable function g with $g(x) \geq f(x)$ for all x.

Theorem 1.1 (Lewis [4])**.** Any hyperimmune-free degree which is not FPF has a strong minimal cover.

Thus any minimal degree which is hyperimmune-free and which constitutes a negative solution to Yates' question must also constitute a positive solution to question 1.1. The fixed point free minimal degree constructed in this paper will be hyperimmune-free.

A second motivation is provided by the interesting techniques required in order to give the positive solution. When $\sigma, \sigma' \in \omega^{<\omega}$ we write $\sigma \star \sigma'$ in order to denote the concatenation of σ and σ'. We write $\sigma|\sigma'$ in order to indicate that these two strings are incompatible and we write $\sigma \subseteq \sigma'$ when σ is an initial segment of σ' . We shall sometimes identify $x \in \omega$ with the corresponding string of length one. In this paper, by a tree we shall mean a partial function $T : \omega^{<\omega} \rightarrow \omega^{<\omega}$ such that for any $\sigma \in \omega^{<\omega}$ and $x \in \omega$, if $T(\sigma \star x) \downarrow$ then:

(i) $T(\sigma) \downarrow$ and $T(\sigma) \subset T(\sigma \star x)$;
(ii) for all $x' < x$, $T(\sigma \star x') \downarrow$ and $T(\sigma \star x')|T(\sigma \star x)$;
(ii) there exists x' such that $T(\sigma \star x') \uparrow$.

We write $\tau \in T$ when τ is in the range of T and we write $f \in [T]$ when there exist an infinite number of initial segments of f in T. We say that τ and τ' are Ψ_m-splitting if $\Psi_m(\tau)|\Psi_m(\tau')$. We say that T is Ψ_m-splitting if whenever $T(\sigma) = \tau$, $T(\sigma') = \tau'$ and $\sigma|\sigma'$ we have $\Psi_m(\tau)|\Psi_m(\tau')$. For nonempty $\Lambda, \Lambda' \subset \omega^{<\omega}$, we say Λ and Λ' are Ψ_m-splitting if whenever $\tau \in \Lambda$ and $\tau' \in \Lambda'$, these two strings are Ψ_m-splitting.

Theorem 1.2 (Lewis [4])**.** Suppose f satisfies the property that whenever $\Psi_m(f)$ is total and $f \leq_T \Psi_m(f)$, f lies on a partial computable Ψ_m-splitting tree. Then f is not of FPF degree.

Thus new techniques are required in order to construct a minimal degree which is FPF. The answer lies in the use of a new kind of splitting tree.

Definition 1.1. We say that a tree T is delayed Ψ_m-splitting if whenever $\tau_0, \tau_1 \in T$ are incompatible, any $\tau_2, \tau_3 \in T$ properly extending τ_0 and τ_1 respectively are Ψ_m-splitting.

Theorem 1.3 (Lewis [4])**.** Any function f is of minimal degree iff f is noncomputable and, whenever $\Psi_m(f)$ is total and noncomputable, f lies on a partial computable delayed Ψ_m-splitting tree.

In order to construct a minimal degree which is FPF, then, we shall have to use delayed splitting trees. This the first example of such a construction in the literature.

In what follows we shall write λ in order to denote the string of length 0. We shall say $\tau \in \omega^{<\omega}$ is DNR if it is an initial segment of a DNR function. For any tree T, we shall say $\tau \in T$ is terminal in T if there does not exist $\tau' \supset \tau$ in T, and we shall say that τ' is a successor of τ in T if there exists $\sigma \in \omega^{<\omega}$ and $x \in \omega$ such that $\tau = T(\sigma)$, $\tau' = T(\sigma \star x)$. We shall use the variables x, n, m, j, k, l, w for elements of ω; i for elements of $\{0,1\}$; σ, τ for elements of $\omega^{<\omega}$; ψ for elements of $2^{<\omega}$; $\Lambda, \Pi, \Delta, \Gamma$ for finite subsets of $\omega^{<\omega}$ and we shall use f, g to denote elements of ω^ω. We say Λ is prefix-free if every two distinct elements of this set are incompatible.

2. The basic framework

If T is a tree then we say that τ is of level l in T if $\tau = T(\sigma)$ for some σ of length l. We say that T is regular if all strings of the same level are of the same length. We say that $T' \subset T$ if the range of T' is a proper subset of the range of T.

Definition 2.1. We say the tree T is Ψ_m-constant if for any l and any τ, τ' of level $l+1$ in T we have that $\Psi_m(\tau; l) \downarrow = \Psi_m(\tau'; l) \downarrow$.

For every m we shall define a partial computable and regular T_m with computable domain. For $m' > m$ we shall have that $T_{m'} \subset T_m$. For $m' > m$ we shall also have that $T_{m'}(\lambda) \supset T_m(\lambda)$ and that $T_{m'}(\lambda)$ is DNR, so that $f = \bigcup_m T_m(\lambda)$ will be a DNR function. In order to ensure that f is of minimal degree we shall ensure that for every $m > 0$ one of three conditions holds.

(i) There exists x such that for no $\tau \in T_m$ is it the case that $\Psi_m(\tau; x) \downarrow$. In this case we shall clearly have that $\Psi_m(f)$ is partial.

(ii) T_m is Ψ_m-constant. In this case we shall clearly have that $\Psi_m(f)$ is computable.

(iii) T_m is delayed Ψ_m-splitting. In this case we shall have that $f \leq_T \Psi_m(f)$. In order to see this suppose we are given an enumeration of T_m and an oracle for $\Psi_m(f)$ and that we wish to compute the initial segment of f of length n. Enumerate T_m until we find $\tau \in T_m$ of level $l+1$ such that $\Psi_m(\tau)$ is compatible with $\Psi_m(f)$ (recall our assumptions regarding Ψ_m) and the initial segment of τ of level l is of length $> n$. The initial segment of τ of length n is an initial segment of f.

3. Bushy sets of strings

We must define each of our trees T_m in such a way that there exists $f \in [T_m]$ which is DNR. In order to achieve this we shall have to use very *bushy* trees. This vague notion is made precise by the following definition, which appeared for the first time in the literature in Kumabe's original proof of the theorem.

Definition 3.1. We say that prefix-free and finite $\Lambda \subseteq \omega^{<\omega}$ is n-*bushy* above $\tau \in \omega^{<\omega}$ if:

(i) there exists $\tau_0 \in \Lambda$ with $\tau_0 \supset \tau$;

(ii) whenever $\tau_1 \in \Lambda$ and $\tau_1 \supset \tau_2 \supseteq \tau$, there exist at least n one element extensions of τ_2 which are initial segments (not necessarily proper) of strings in Λ.

In order to see clearly what definition 3.1 means, suppose that Λ is a prefix-free and finite set of strings with at least one element properly extending τ, and that we wish to choose $\tau_0 \in \Lambda$ extending τ by defining this string in a finite sequence of stages. Beginning with τ we decide at each stage what the next bit of τ_0 should be, with the restriction always that the resulting string should have an extension in Λ, until after a finite number of stages we arrive at a string in Λ. Λ being n-bushy above τ means that at every stage in this process we shall have at least n choices as to how to define the next bit of τ_0, no matter how we have proceeded beforehand. Of course, the crucial point here is that if τ is DNR and Λ is 2-bushy above τ, then we can choose τ_0 in Λ properly extending τ and which is also DNR. If $T(\lambda)$ is DNR, T is without terminal strings and whenever $T(\sigma)\downarrow$ the finite set $\{T(\sigma \star k) : k \in \omega,\ T(\sigma \star k)\downarrow\}$ is 2-bushy above $T(\sigma)$, then there exists $f \in [T]$ which is DNR.

Bushy sets of strings satisfy some easily verifiable properties which will be useful to us later.

Lemma 3.1. Let prefix-free and finite Λ be n-bushy above τ.

(i) (The concatenation property.) For each $\tau_0 \in \Lambda$ suppose that Λ_{τ_0} is either equal to $\{\tau_0\}$ or is a prefix-free and finite set of strings extending τ_0 which is n-bushy above τ_0. Then $\bigcup_{\tau_0 \in \Lambda} \Lambda_{\tau_0}$ is n-bushy above τ.

(ii) (The sparse subset property.) For some nonzero $n' \leq n$ and $\Lambda' \subseteq \Lambda$ suppose that no subset of Λ' is n'-bushy above τ. Then there exists some subset of $\Lambda - \Lambda'$ which is $(n - n' + 1)$-bushy above τ.

(iii) (The second sparse subset property.) Suppose no subset of $\Lambda_0 \subseteq \Lambda$ is n_0-bushy above τ, that no subset of $\Lambda_1 \subseteq \Lambda$ is n_1-bushy above τ and that $n_0, n_1 > 0$. Then there is no subset of $\Lambda_0 \cup \Lambda_1$ which is $(n_0 + n_1 - 1)$-bushy above τ.

Proof. (i) is obvious. We prove (ii) by induction on l which is the maximum value $|\tau'| - |\tau|$ such that $\tau' \in \Lambda$ and extends τ. For $l = 1$ the result is immediate. Now suppose that $l > 1$. There can be at most $n' - 1$ one element extensions τ' of τ which are either in Λ' or for which there exists some subset of Λ' which is n'-bushy above τ'. By the induction hypothesis there therefore exist at least $n - n' + 1$ one element extensions τ' of τ which are either in $\Lambda - \Lambda'$ or for which there exists some subset of $\Lambda - \Lambda'$ which is $(n - n' + 1)$-bushy above τ'.

We prove (iii) similarly. In order to see the induction step observe that there can be at most $n_0 - 1$ one element extensions τ' of τ which are either in Λ_0 or for which there exists some subset of Λ_0 which is n_0-bushy above τ'. In exactly the same way, there can be at most $n_1 - 1$ one element extensions τ' of τ which are either in Λ_1 or for which there exists some subset of Λ_1 which is n_1-bushy above τ'. Thus, by the induction hypothesis, there can exist at most $n_0 + n_1 - 2$ one element extensions of τ which are either in $\Lambda_0 \cup \Lambda_1$ or for which there exists some subset of $\Lambda_0 \cup \Lambda_1$ which is $(n_0 + n_1 - 1)$-bushy above τ'. □

Roughly speaking the sparse subset property says that if Λ is bushy and $\Lambda' \subset \Lambda$ is not, then $\Lambda - \Lambda'$ is bushy. The second sparse subset property says that if Λ_0 and Λ_1 are not bushy then neither is their union. Hopefully it is clear the sort of use that we shall be able to make of these properties. If Λ is $2n$-bushy above τ, for example, then the sparse subset property tells us that for any Turing functional Ψ, either we can find $\Lambda' \subset \Lambda$ which is n-bushy above τ and such that for all $\tau' \in \Lambda'$ we have $\Psi(\tau'; |\Psi(\tau)|) \downarrow$, or we can find $\Lambda' \subset \Lambda$ which is n-bushy above τ and such that for all $\tau' \in \Lambda'$ we have $\Psi(\tau'; |\Psi(\tau)|) \uparrow$.

The basic idea, then, is that we shall enumerate bushy sets of strings into our trees. For each m we shall define a computable and increasing function g_m which specifies the level of bushiness required at each level of T_m (by increasing we mean strictly increasing). Whenever we go to define the successors of $T_m(\sigma)$ in T_m we shall ensure that this set $\{T_m(\sigma \star k) : k \in \omega,\ T_m(\sigma \star k) \downarrow\}$ is $g_m(|\sigma|)$-bushy above $T_m(\sigma)$.

There is a slight complication that has to be considered before we go on to describe how we can go about defining any T_{m+1} given T_m. We observed previously that if $T(\lambda)$ is DNR, T is without terminal strings and whenever $T(\sigma) \downarrow$ the finite set $\{T(\sigma \star k) : k \in \omega,\ T(\sigma \star k) \downarrow\}$ is 2-bushy above $T(\sigma)$, then there exists $f \in [T]$ which is DNR. We shall not actually be able to insist, however, that the T_m are without terminal strings. Thus $g_m(|\sigma|)$ will be considered to specify the required bushiness for the successors of $T_m(\sigma)$ only in the case that any such successors exist. What can we do in this situation in order to ensure that there exists $f \in [T_m]$ which is DNR? It suffices that $T_m(\lambda)$ is DNR and not terminal, g_m is increasing, that $g_m(0) \geq 2$ and that for some nonzero $w_m < g_m(0)$ there should be no (prefix free and finite) set of terminal strings on T_m which is w_m-bushy above $T_m(\lambda)$. In order to see this suppose given $l > 0$. Let Λ be the set of strings in T_m which are either of level l, or else of level $l' < l$ and terminal in T_m. Since g_m is increasing, Λ is $g_m(0)$-bushy above $T_m(\lambda)$. Let Λ' be the set of all strings in Λ which are terminal in T_m. Since no subset of Λ' is w_m-bushy above $T_m(\lambda)$, it follows by the sparse subset property that the set of strings in Λ which are not terminal in T_m has a subset which is $(g_m(0) - w_m + 1)$-bushy above $T_m(\lambda)$. Since $g_m(0) - w_m + 1 \geq 2$, it follows that for each l there exists τ of level l in T_m and which is DNR. For each m, then, we shall also define a value $w_m \geq 1$ and we shall ensure that there is no set of terminal strings on T_m which is w_m-bushy above $T_m(\lambda)$.

To finish this section let us summarize those restrictions that we have so far placed on each g_m and w_m:

(R1) g_m is increasing (i.e. strictly increasing);
(R2) $1 \leq w_m < g_m(0)$.

As we continue to describe the manner in which the construction operates, we will list further restrictions to be placed on these values. Once all such restrictions have been established we will then easily be able to define these values so as to meet all the restrictions we have listed.

4. Forcing $\Psi_{m+1}(f)$ to be partial

Let us suppose that we have already defined T_m and that this is a partial computable and regular tree with computable domain. We have defined already g_m and w_m satisfying (R1) and (R2), such that g_m is computable and specifies the bushiness at each level of T_m as previously described. There does not exist a set of terminal strings on T_m which is w_m-bushy above $T_m(\lambda)$. We have also that $T_m(\lambda)$ is DNR and not terminal.

Which strings on T_m are suitable values for $T_{m+1}(\lambda)$? We require at least that $T_{m+1}(\lambda)$ should be DNR and we also require that there should exist f extending $T_{m+1}(\lambda)$ in $[T_m]$ which is DNR. It is these considerations which motivate the following definition.

Definition 4.1. We say that $B(m, \tau)$ holds if all of the following are true:

(i) $\tau = T_m(\sigma)$ for some σ of length ≥ 1;
(ii) τ is DNR;
(iii) τ is not terminal on T_m and there does not exist a set of terminal strings on T_m which is w_m-bushy above τ.

The discussion of the previous section suffices to show that if $B(m, \tau)$ holds then there exists f extending τ in $[T_m]$ and which is DNR. Before defining $T_{m+1}(\lambda) = \tau$, then, we shall require that $B(m, \tau)$ holds.

Note also, that if $B(m, \tau)$ holds for some τ of level l in T_m, then for each $l' > l$ we can find $\tau' \supset \tau$ of level l' and such that $B(m, \tau')$ holds. In order to see this, suppose given $l' > l$. Let Λ be the set of strings extending τ and which are either of level l' in T_m, or of level $l'' < l'$ and terminal in T_m. Since g_m is increasing, Λ is $g_m(0)$-bushy above τ. Now consider Λ' which is the set of those strings $\tau' \in \Lambda$ such that $B(m, \tau')$ does not hold. There are two possible reasons why $B(m, \tau')$ does not hold. Firstly, it may be the case that τ' is terminal on T_m or that there exists a set of terminal strings on T_m which is w_m-bushy above τ'. Since $B(m, \tau)$ holds, it follows from the concatenation property that no set of such strings is w_m-bushy above τ. Otherwise it must be the case that τ' is not DNR. Since τ is DNR, no set of such strings is 2-bushy above τ. By the second sparse subset it follows that Λ' is not $g_m(0)$-bushy above τ. Therefore $\Lambda - \Lambda'$ is non-empty. Any element of this set must be of level l' in T_m.

The first option we consider in looking to define T_{m+1} is that we may be able to define this tree so as to force $\Psi_{m+1}(f)$ to be partial. We can do this if there exists $\tau^\star$ on T_m such that $B(m, \tau^\star)$ holds and such that there does not exist a bushy set of strings on T_m extending $\tau^\star$, Λ say, such that for all $\tau \in \Lambda$, $\Psi_{m+1}(\tau; |\Psi_{m+1}(\tau^\star)|) \downarrow$. More specifically, we ask (Q1) below, and if this question receives a positive answer then we build T_{m+1} so as to force $\Psi_{m+1}(f)$ to be partial. Numerical values used here are somewhat arbitrary—the important thing is that, as will be explained, they suffice for our purposes.

(Q1) Does there exist $\tau^\star = T_m(\sigma^\star)$ such that, letting $n^\star = g_m(|\sigma^\star|)/4$:
   (a) $B(m, \tau^\star)$ holds;
   (b) $w_m < n^\star$;
   (c) there does not exist an $n^\star$-bushy set of strings on T_m above $\tau^\star$, Λ say, such that for all $\tau \in \Lambda$, $\Psi_{m+1}(\tau, |\Psi_{m+1}(\tau^\star)|) \downarrow$?

In order that (Q1) should make sense we have implicitly required:

(R3)′ $g_m(l)$ is a positive multiple of 4 if $l \geq 1$.

In fact, since (R1) and (R2) are satisfied, (R3) below suffices to ensure satisfaction of (R3)′.

(R3) $g_m(l)$ is a power of 2.

Note also that, since g_m is increasing, (b) of (Q1) requires only that $\sigma^\star$ should be of sufficient length. In fact, since (R1)–(R3) are satisfied, satisfaction of (b) requires only that $\sigma^\star$ should be of length at least 2.

Definition 4.2. For any $m, l \in \omega$ we let $\kappa_{m+1}(l)$ be that unique l' such that all strings of level l in T_{m+1} are of level l' in T_m.

So suppose that (Q1) receives a positive answer. Then we define $T_{m+1}(\lambda) = \tau^\star$ and $g_{m+1}(0) = 2n^\star$. We shall not yet specify exactly how g_{m+1} should be defined on arguments > 0. We shall leave that task until all restrictions on how we should do so have been established. Instead, we suppose that we *do* know already how g_{m+1} should be defined, and we describe how to define T_{m+1} in this case. For now we note only that, together with (R1)–(R3), g_{m+1} will satisfy:

(R4) for all $l \geq 1$, $g_{m+1}(l) \leq g_m(\kappa_{m+1}(l))/2$.

Why we require satisfaction of a condition like (R4) (once again this condition is somewhat arbitrary) will be clear from the analysis given subsequent to formally defining T_{m+1}. The basic idea behind the construction of T_{m+1} is as follows. Each string $\tau = T_{m+1}(\sigma)$ is also in T_m. It may be the case (when $\tau \neq \tau^\star$) that τ is terminal in T_m, but since $B(m, \tau^\star)$ holds and $w_m < n^\star$ there cannot exist a set of strings of this kind which is $n^\star$-bushy above $\tau^\star$. If τ is not terminal in T_m then we can look amongst the successors of τ in T_m for a set of strings which is $g_{m+1}(|\sigma|)$-bushy above τ and such that every member τ' satisfies $\Psi_{m+1}(\tau', |\Psi_{m+1}(\tau^\star)|) \uparrow$. If there exists such, then we define these to be the successors of τ in T_{m+1}. If not, then we make τ terminal in T_{m+1}. The only difficulty here, then, is to show (roughly speaking) that if we proceed in this way, we shall not have a very bushy set of terminal strings in T_{m+1}. More precisely, we must show that we can define a value for w_{m+1} which is less than $g_{m+1}(0)$. We shall explain how and why we can do this after giving the formal definition of T_{m+1}.

Defining $\mathbf{T_{m+1}}$. We define $T_{m+1}(\lambda) = \tau^\star$. Suppose that we have defined $T_{m+1}(\sigma)$ and let $T_{m+1}(\sigma) = T_m(\sigma')$. Next we ask:

Does there exist Λ which is $g_{m+1}(|\sigma|)$-bushy above $T_{m+1}(\sigma)$, which is a subset of $\{T_m(\sigma' \star k) : k \in \omega, T_m(\sigma' \star k) \downarrow\}$, and such that for every $\tau' \in \Lambda$, $\Psi_{m+1}(\tau'; |\Psi_{m+1}(\tau^\star)|) \uparrow$?

If so then let $\Lambda = \{\tau_0, .., \tau_k\}$ and for all $k' \leq k$ define $T_{m+1}(\sigma \star k') = \tau_{k'}$, leaving $T_{m+1}(\sigma \star k')$ undefined for all $k' > k$. Otherwise leave $T_{m+1}(\sigma \star k')$ undefined for all k'.

A number of things should be immediately clear about T_{m+1} as defined above. Firstly, T_{m+1} is partial computable and regular with computable domain. That T_{m+1} has computable domain follows from the fact that T_m has computable domain and the convention, stated previously, that for any $\tau \in \omega^{<\omega}$ and any $x \in \omega$, $\Psi_{m+1}(\tau; x)$ is defined only if this computation converges in $< |\tau|$ many steps and $\Psi_{m+1}(\tau; x')$ is defined for all $x' < x$. That T_{m+1} is regular follows from the fact that for every l there exists l' such that every string of level l in T_{m+1} is of level l' in T_m, and the fact that T_m is regular. Secondly, the function g_{m+1} really does specify the bushiness at each level of T_{m+1} in the way that it is supposed to. If we define any successors of $T_{m+1}(\sigma)$ we have ensured that this set $\{T_{m+1}(\sigma \star k) : k \in \omega,\ T_{m+1}(\sigma \star k) \downarrow\}$ is $g_{m+1}(|\sigma|)$-bushy above $T_{m+1}(\sigma)$. Thirdly it is clear that for any $\tau \in T_{m+1}$ we have $\Psi_{m+1}(\tau; |\Psi_{m+1}(\tau^\star)|) \uparrow$. Lastly, it is also clear that $T_{m+1}(\lambda)$ is DNR and it follows from the sparse subset property that this string is not terminal in T_{m+1}.

What we are left to show is that we can define an appropriate value for w_{m+1}. In order to see this consider the set of strings which are terminal in T_{m+1}. If $\tau = T_{m+1}(\sigma) = T_m(\sigma')$ is terminal in T_{m+1} then this must be for one of two reasons. It may be that τ is terminal in T_m. As observed previously, we chose $\tau^\star$ in such a way as to ensure there is no $n^\star$-bushy set of such strings above $\tau^\star$. If τ is terminal in T_{m+1} but is not terminal in T_m then by the sparse subset property, and since (R4) is satisfied by g_{m+1}, there exists a set of successors of τ in T_m, which is $(g_m(|\sigma'|)/2)$-bushy above τ and such that for every τ' in this set $\Psi_{m+1}(\tau'; |\Psi_{m+1}(\tau^\star)|) \downarrow$. Since $n^\star = g_m(|\sigma^\star|)/4$ and g_m is increasing, we conclude that there exists a set of successors of τ in T_m, which is $n^\star$-bushy above τ and such that for every τ' in this set $\Psi_{m+1}(\tau'; |\Psi_{m+1}(\tau^\star)|) \downarrow$. Thus, by the concatenation property and since $\tau^\star$ satisfies (c) of (Q1), there exists no set of strings which are terminal in T_{m+1} but not terminal in T_m, and which is $n^\star$-bushy above $\tau^\star$. Applying the second sparse subset property to the sets of strings which are terminal in T_{m+1} for the two reasons stated above, we conclude that there does not exist a set of terminal strings in T_{m+1} which is $(2n^\star - 1)$-bushy above $\tau^\star$. We therefore define $w_{m+1} = 2n^\star - 1 = g_{m+1}(0) - 1$.

5. Forcing $\Psi_{m+1}(f)$ to be computable

Next suppose that (Q1) receives a negative response. We remarked previously that satisfaction of (b) of (Q1) requires only that $\sigma^\star$ should be of length at least 2. Thus the fact that (Q1) receives a negative response means that for any $\tau \in T_m$ of level $l \geq 2$ such that $B(m, \tau)$ holds, there exists a set of strings on T_m, Λ say, which is $(g_m(l)/4)$-bushy above τ and such that for every $\tau' \in \Lambda$, $\Psi_{m+1}(\tau'; |\Psi_{m+1}(\tau)|) \downarrow$.

The next possibility we consider is that we can force $\Psi_{m+1}(f)$ to be computable. We can do this if (Q2) below receives a positive response. Once again numerical values used here are somewhat arbitrary.

(Q2) Does there exist $\tau^\star = T_m(\sigma^\star)$ with $|\sigma^\star| \geq 2$ such that, letting $n^\star = g_m(|\sigma^\star|)/64$:
  (a) $B(m, \tau^\star)$ holds;
  (b) $w_m < n^\star$;
  (c) there does not exist Λ_0, Λ_1 which are $n^\star$-bushy sets of strings on T_m above $\tau^\star$, and which are Ψ_{m+1}-splitting?

By almost precisely the argument we gave subsequent to definition 4.1, satisfaction of (a) and (b) suffices to ensure that there does not exist a set of strings on T_m, which is $n^\star$-bushy above $\tau^\star$ and such that for every τ in this set, $B(m, \tau)$ does not hold.

So suppose that (Q2) receives a positive answer. Then we define $T_{m+1}(\lambda) = \tau^\star$ and $g_{m+1}(0) = 4n^\star$. Once again, we shall not specify precisely at this point how g_{m+1} should be defined on arguments > 0. We leave this task until all restrictions upon how we must make this definition have been established. We suppose we do know how g_{m+1} is to be defined and we describe how to define T_{m+1} in this case. For now we note only that, together with (R1)–(R4), g_{m+1} will satisfy:

(R5) for all $l \geq 1$, $g_{m+1}(l) \leq g_m(\kappa_{m+1}(l))/16$.

Why we require a condition along these lines will become clear from the discussion that follows. While defining T_{m+1} we let Π^l denote the set of strings of the form $\tau = T_{m+1}(\sigma)$ for σ of length $< l$ which are terminal in T_{m+1} and such that $B(m, \tau)$ holds.

Defining $\mathbf{T_{m+1}}$. Initially we define $T_{m+1}(\lambda) = \tau^\star$. Now suppose that for all σ of length $\leq l$ we have either defined $T_{m+1}(\sigma)$ or we have decided that this value should be undefined. Let Γ be the set of strings of the form $\tau = T_{m+1}(\sigma)$ such that either σ is of length $< l$ and τ is terminal in T_{m+1}, or such that σ is of length l. Assume inductively that:

(i) Π^l has no subset which is $n^\star$-bushy above $\tau^\star$;
(ii) for each $\tau \in \Pi^l$ we can choose a set of strings on T_m which is $n^\star$-bushy above τ and such that each τ' in this set and any τ'' of level l in T_{m+1} are Ψ_{m+1}-splitting;
(iii) Γ is $4n^\star$-bushy above $\tau^\star$ if $l \geq 1$;
(iv) for all τ of level l in T_{m+1} we have $|\Psi_{m+1}(\tau)| \geq l$.

Step 1. For each $\tau = T_{m+1}(\sigma)$ for σ of length l search until either it is found that $B(m, \tau)$ does not hold or until we find a $4g_{m+1}(l)$-bushy set of strings on T_m above τ, Λ_τ say, such that for all $\tau' \in \Lambda_\tau$, $\Psi_{m+1}(\tau'; |\Psi_{m+1}(\tau)|) \downarrow$. As remarked at the beginning of this section, and since (R5) is satisfied, at least one of these two possibilities must hold. If $\Lambda_\tau \downarrow$ then, by the sparse subset property, we can find $\Lambda'_\tau \subset \Lambda_\tau$ which is $2g_{m+1}(l)$-bushy above τ and such that for all τ' in this set $\Psi_{m+1}(\tau';l)$ takes the same value, i_τ say.

Now we must take a little care in order to ensure that T_{m+1} remains regular. Let $l^\star$ be longer than any string in any Λ'_τ. For each τ such that $\Lambda'_\tau \downarrow$ search until either it is found that $B(m,\tau)$ does not hold or until we find Λ''_τ such that each string in this set is an extension of a string in Λ'_τ, which is a $g_{m+1}(l)$-bushy set of strings on T_m above τ and such that every string in this set is of length $l^\star$. We can do this since:

(a) if $B(m,\tau)$ holds, there does not exist a subset of Λ'_τ such that for every string τ' in this set $B(m,\tau')$ does not hold and which is $g_{m+1}(l)$-bushy above τ;
(b) if $B(m,\tau')$ holds for $\tau' \in \Lambda'_\tau$ there exists a $g_{m+1}(l)$-bushy set of strings of length $l^\star$ on T_m above τ'.

If $l=0$ then let $\Lambda''_{\tau^\star} = \{\tau_0,..,\tau_k\}$. For all $k' \leq k$ define $T_{m+1}(k') = \tau_{k'}$, leaving $T_{m+1}(k')$ undefined for all $k' > k$. Clearly all parts of the induction hypothesis now hold for $l=1$.

If $l>0$ proceed to step 2.

Step 2. Let Λ_i be the set of all τ of level l in T_{m+1} for which $\Lambda''_\tau \downarrow$ and $i_\tau = i$. Let $\Lambda = \Lambda_0 \cup \Lambda_1$. By previous observation and the induction hypothesis:

(1) Γ is $4n^\star$-bushy above $\tau^\star$;
(2) Π^l has no subset which is $n^\star$-bushy above $\tau^\star$;
(3) there does not exist a set of strings on T_m, which is $n^\star$-bushy above $\tau^\star$ and such that for all τ in this set $B(m,\tau)$ does not hold.

It therefore follows by the sparse subset properties that there exists a subset of Λ which is $2n^\star$-bushy above $\tau^\star$. By the sparse subset property we may therefore fix i such that there exists a subset of Λ_i which is $n^\star$-bushy above $\tau^\star$. For each $\tau = T_{m+1}(\sigma)$ in Λ_i proceed as follows. Let $\Lambda''_\tau = \{\tau_0,..,\tau_k\}$ and for all $k' \leq k$ define $T_{m+1}(\sigma \star k') = \tau_{k'}$, leaving $T_{m+1}(\sigma \star k')$ undefined for all $k' > k$. Any τ of level l in T_{m+1} and which is not in Λ_i has no successors in T_{m+1}.

Let Γ' be the set of strings of the form $\tau = T_{m+1}(\sigma)$ such that either σ is of length $< l+1$ and τ is terminal in T_{m+1}, or such that σ is of length $l+1$. Since g_{m+1} is increasing, it follows immediately from the concatenation property that Γ' is $4n^\star$-bushy above $\tau^\star$. Let Γ'' be the set of strings of the form $\tau = T_{m+1}(\sigma)$ such that σ is of length $l+1$. By the way in which we chose i above, Γ'' has a subset which is $n^\star$-bushy. For each $\tau \in \Pi^{l+1}$ we can choose a set of strings on T_m which is $n^\star$-bushy above τ and such that for each τ' in this set and any $\tau'' \in \Gamma''$, $\Psi_{m+1}(\tau')$ and $\Psi_{m+1}(\tau'')$ are incompatible. Since $\tau^\star$ satisfies (c) of (Q2) it follows that no subset of Π^{l+1} is $n^\star$-bushy above $\tau^\star$. If $\tau \in \Gamma''$ then $|\Psi_{m+1}(\tau)| \geq l+1$.

It is clear that T_{m+1} defined in this way is Ψ_{m+1}-constant, partial computable and regular with computable domain. $T_{m+1}(\lambda)$ is DNR and is not terminal in T_{m+1}. Any string τ which is terminal in T_{m+1} is either in Π^l

for some l or is such that $B(m,\tau)$ does not hold. We may therefore define $w_{m+1} = 2n^\star - 1$.

6. Forcing $f \leq_T \Psi_{m+1}(f)$

The last case we have to consider is that both (Q1) and (Q2) receive a negative answer. In this case we must define T_{m+1} to be a delayed Ψ_{m+1}-splitting tree. Before describing how to do so, a couple of lemmas are required. These two lemmas will provide the basic tools with which we shall be able to build up bushy sets of strings which are Ψ_{m+1}-splitting and which we can enumerate into T_{m+1}. As before, numerical values used in what follows are somewhat arbitrary and are chosen in such a way as to make calculations as simple as possible.

Lemma 6.1. Suppose that (Q1) receives a negative answer, that $\tau = T_m(\sigma)$, $8w_m < g_m(|\sigma|)$ and that $B(m,\tau)$ holds. Then for any $l \in \omega$ there exists a set of strings on T_m, Λ say, which is $\frac{1}{8}g_m(|\sigma|)$-bushy above τ and such that for all $\tau' \in \Lambda$, $|\Psi_{m+1}(\tau')| > l$ and $B(m,\tau')$ holds.

Proof. The proof is by induction on l. First consider the case $l = |\Psi_{m+1}(\tau)|$. Since (Q1) receives a negative response, $B(m,\tau)$ holds and $w_m < g_m(|\sigma|)/4$ there exists a set of strings on T_m, Λ' say, which is $\frac{1}{4}g_m(|\sigma|)$-bushy above τ and such that for all $\tau' \in \Lambda'$, $|\Psi_{m+1}(\tau')| > l$. Since there does not exist Λ'' which is a set of strings on T_m, which is $\frac{1}{8}g_m(|\sigma|)$-bushy above τ and such that for all $\tau' \in \Lambda''$, $B(m,\tau')$ does not hold, it follows from the sparse subset property that there exists $\Lambda \subset \Lambda'$ as required.

Now suppose that there exists Λ' which is a set of strings on T_m, which is $\frac{1}{8}g_m(|\sigma|)$-bushy above τ and such that for all $\tau' \in \Lambda'$, $|\Psi_{m+1}(\tau')| > l$ and $B(m,\tau')$ holds. For each $\tau' \in \Lambda'$ it follows by exactly the same reasoning that we used for the base case above (and since g_m is increasing), that there exists Λ'' which is a set of strings on T_m, which is $\frac{1}{8}g_m(|\sigma|)$-bushy above τ' and such that for all $\tau'' \in \Lambda''$, $|\Psi_{m+1}(\tau'')| > l+1$ and $B(m,\tau'')$ holds. The existence of Λ sufficient to complete the induction step follows immediately from the concatenation property. □

Lemma 6.2. Suppose that n_0 and n_1 are positive multiples of four. Let Λ be a prefix-free and finite set of strings on T_m which is n_0-bushy above $\tau_0 \in T_m$. For each $\tau \in \Lambda$ and $i \in \{0,1\}$ suppose that there exists $\Lambda_{i,\tau}$ which is a prefix-free and finite set of strings on T_m extending τ, which is n_0-bushy above τ and such that $\Lambda_{0,\tau}$ and $\Lambda_{1,\tau}$ are Ψ_{m+1}-splitting. Let $\Lambda'' = \bigcup_{i,\tau} \Lambda_{i,\tau}$. Let Π be a prefix-free and finite set of strings on T_m, which is n_1-bushy above $\tau_1 \in T_m$ and such that for any $\tau' \in \Pi$ and any $\tau \in \Lambda''$, $|\Psi_{m+1}(\tau')| > |\Psi_{m+1}(\tau)|$. Then there exist $\Lambda' \subset \Lambda''$ and $\Pi' \subset \Pi$ such that:

(i) Λ' is $n_0/4$-bushy above τ_0;
(ii) Π' is $n_1/4$-bushy above τ_1;
(iii) Λ' and Π' are Ψ_{m+1}-splitting.

Proof. Note first that Λ'' is n_0-bushy above τ_0. Let v be the least value $|\Psi_{m+1}(\tau)|$ such that $\tau \in \Pi$ and note that $v > |\Psi_{m+1}(\tau')|$ for any $\tau' \in \Lambda''$. We define a finite sequence of binary strings $\{\psi_k\}$ such that for each k, $\psi_{k+1} \supset \psi_k$ if $\psi_{k+1} \downarrow$, and:

(†) the set Π^0_k of strings $\tau \in \Pi$ such that $\Psi_{m+1}(\tau) \supseteq \psi_k$ has a subset which is $n_1/4$-bushy above τ_1.

We define ψ_0 to be the longest string which is an initial segment of every $\Psi_{m+1}(\tau)$ for $\tau \in \Pi$. Now suppose that we have defined ψ_k. We make the following definitions.

(a) We let Π^1_k be the set of strings $\tau \in \Pi$ such that $\Psi_{m+1}(\tau)|\psi_k$.
(b) We let Λ^0_k be the set of strings $\tau \in \Lambda''$ such that $\Psi_{m+1}(\tau)|\psi_k$.
(c) We let Λ^1_k be the set of strings $\tau \in \Lambda''$ such that $\Psi_{m+1}(\tau) \subseteq \psi_k$.
(d) We let Λ^2_k be the set of strings $\tau \in \Lambda''$ such that $\Psi_{m+1}(\tau) \supset \psi_k$.

Since $\Lambda'' = \Lambda^0_k \cup \Lambda^1_k \cup \Lambda^2_k$ at least one of these three sets has a subset which is $n_0/4$-bushy above τ_0. There are four cases to consider.

Case 1. Λ^0_k has a subset which is $n_0/4$-bushy above τ_0. Then the lemma follows immediately since (†) is satisfied.

Case 2. Λ^1_k has a subset $\Lambda^\star$ which is $n_0/4$-bushy above τ_0. Then let Γ be the set of all $\tau \in \Lambda$ which have an extension in $\Lambda^\star$. For each $\tau \in \Gamma$ there exists i such that every extension of τ in $\Lambda^\star$ is in $\Lambda_{i,\tau}$. Let $i_\tau = 1 - i$ and let $\Lambda' = \bigcup_{\tau \in \Gamma} \Lambda_{i_\tau,\tau}$. The lemma follows immediately since Λ' is $n_0/4$-bushy above τ_0 and is Ψ_{m+1}-splitting with Π' whose existence is ensured by the satisfaction of (†).

Case 3. Neither case 1 nor case 2 apply but Π^1_k has a subset which is $n_1/4$-bushy above τ_1. In this case the lemma follows immediately since Λ^2_k has a subset which is $n_0/4$-bushy above τ_0.

Case 4. None of the above. We show first that in this case $|\psi_k| < v$. This suffices to prove that the sequence $\{\psi_k\}$ is finite. So suppose that $|\psi_k| \geq v$. Then $\Lambda'' = \Lambda^0_k \cup \Lambda^1_k$ so that one of cases 1 or 2 must apply, a contradiction.

Since Π^1_k has no subset which is $n_1/4$-bushy above τ_1, Π^0_k has a subset which is $n_1/2$-bushy above τ_1. By the sparse subset property we may therefore choose ψ_{k+1} to be a one element extension of ψ_k and such that Π^0_{k+1} has a subset which is $n_1/4$-bushy above τ_1.

□

Definition 6.1. For any tree T we let $w(T, l)$ denote the number of strings which are either of level l in T, or else of level $l' < l$ in T and terminal in T. Note that for any tree T, $w(T, l)$ is nondecreasing as a function of l.

The fact that (Q2) receives a negative response means that, for any $\tau^\star = T_m(\sigma^\star)$ with $|\sigma^\star| \geq 2$ such that, letting $n^\star = g_m(|\sigma^\star|)/64$,

(1) $B(m, \tau^\star)$ holds and
(2) $w_m < n^\star$,

there exist Λ_0, Λ_1 which are $n^\star$-bushy sets of strings on T_m above $\tau^\star$, and which are Ψ_{m+1}-splitting. In order to define T_{m+1}, then, we first choose such $\tau^\star$. The fact that we can choose such a $\tau^\star$ follows from the discussion subsequent to definition 4.1. We define $T_{m+1}(\lambda) = \tau^\star$, we define the strings of level 1 in T_{m+1} to be the successors of $\tau^\star$ in T_m and we define $g_{m+1}(0) = g_m(|\sigma^\star|)$. We shall not yet specify precisely how g_{m+1} should be defined on all remaining values. As before, we suppose that we do know how this function is to be defined and we describe how to define T_{m+1} in this case. For now we note only that, together with (R1)–(R5), the following condition will be satisfied:

(R6) For all $l \geq 1$, $g_{m+1}(l) \leq g_m(\kappa_{m+1}(l))/(2^{5+3w(T_{m+1},l)})$.

Presumably the reader will initially have no idea why we should want satisfaction of a condition like (R6). The reasons will become clear from what follows.

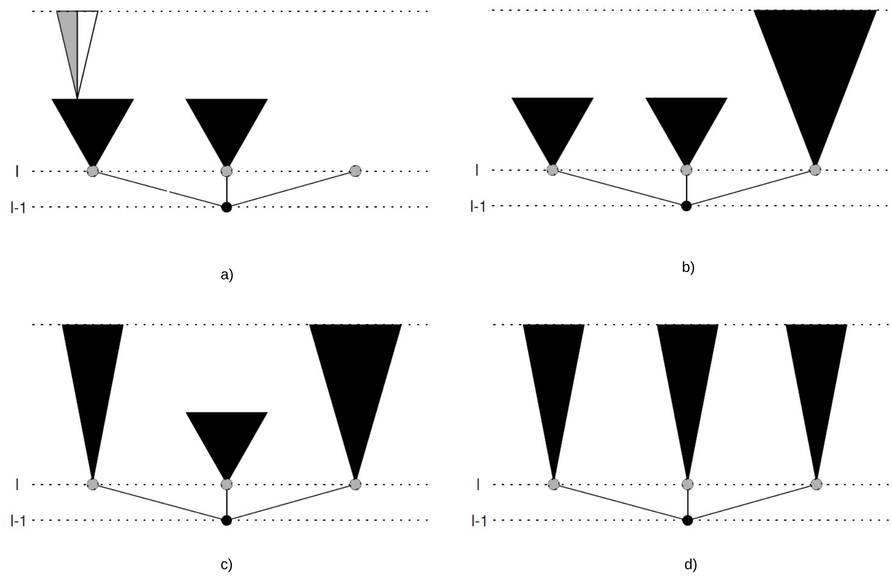

Figure 1.

Defining $\mathbf{T_{m+1}}$. For $l \geq 1$ suppose that for all σ of length $\leq l$ we have already defined $T_{m+1}(\sigma)$ or we have decided that this value should be undefined. Let $l' = \kappa_{m+1}(l)$.

Step 1. For each τ of level $l-1$ in T_{m+1} which has successors in T_{m+1} proceed as follows. Let $\Gamma = \{\tau_0, .., \tau_k\}$ be the successors of τ in T_{m+1} and note that $k < w(T_{m+1}, l)$. For each of these strings τ_j $(0 \leq j \leq k)$ begin running a search which will terminate if $B(m, \tau_j)$ does not hold. If it is found that $B(m, \tau_j)$ does not hold before step 1 terminates then remove τ_j from Γ and proceed as if this string had never been in Γ. We run the following $k+1$ step iteration. (As strings are removed from Γ it may turn out that this iteration actually consists of less than $k+1$ steps.)

Iteration step 0. Let $\Lambda^0_{\tau_0}$ be a $g_m(l')/2^5$-bushy set of strings above τ_0 on T_m. The fact that such $\Lambda^0_{\tau_0}$ exists follows from the fact that $g_{m+1}(0) \geq 2$, g_{m+1} is increasing, (R6) is satisfied and the fact that we may assume τ_0 is not terminal in T_m, otherwise this string will be removed from Γ.

Before describing step $k'+1$ precisely, we give a rough idea of the procedure. By the end of step k' we will already have defined sets $\Lambda^{k'}_{\tau_j}$ for each $j \leq k'$, such that $\Lambda^{k'}_{\tau_j}$ and $\Lambda^{k'}_{\tau_{j'}}$ are Ψ_{m+1}-splitting when $j \neq j'$. First we find a bushy subset of each $\Lambda^{k'}_{\tau_j}$ such that above each string τ' in this subset, there exist two bushy sets of strings which are Ψ_{m+1}-splitting. The diagram (a) gives a rough illustration of the situation when $k' = 1$. Then we apply Lemma 6.1 in order to get a bushy set of strings above $\tau_{k'+1}$ such that $\Psi_{m+1}(\tau')$ is very long for every string τ' in this set, as illustrated in (b). We then apply Lemma 6.2 in order to get bushy sets of strings above τ_0 and $\tau_{k'+1}$ which are Ψ_{m+1}-splitting, as illustrated in (c). Then we apply Lemma 6.2 in order to get bushy sets of strings above τ_1 and $\tau_{k'+1}$ which are Ψ_{m+1}-splitting, as illustrated in (d), and so on. The point of (R6) is simply that it allows us to make all of these successive applications of Lemma 6.2 and still end up with a set of strings above each τ_j which is sufficiently bushy.

Iteration step $k'+1$ for $0 \leq k' < k$. We have already defined sets $\Lambda^{k'}_{\tau_j}$ for each $j \leq k'$. Putting $n_{k'} = g_m(l')/(2^{5+3(k')})$ we have that each of these $\Lambda^{k'}_{\tau_j}$ is $n_{k'}$-bushy above τ_j on T_m and we can therefore find some $\Delta^{k'}_{\tau_j} \subset \Lambda^{k'}_{\tau_j}$ which is $n_{k'}/2$-bushy above τ_j and such that for every string $\tau' \in \Delta^{k'}_{\tau_j}$ we have some $\Lambda_{0,\tau'}$ and $\Lambda_{1,\tau'}$ which are $n_{k'}/2$-bushy above τ' on T_m and which are Ψ_{m+1}-splitting. In order to see this observe first that we may assume that $B(m, \tau_j)$ holds, otherwise τ_j will eventually be removed from Γ in the case that no such $\Delta^{k'}_{\tau_j}$ exists. The fact that we can find $\Delta^{k'}_{\tau_j}$ as required therefore follows from the sparse subset property and the fact that:

(1) there does not exist an $n^\star$-bushy set of strings on T_m above τ_j and such that for all τ' in this set $B(m, \tau')$ does not hold;
(2) since g_{m+1} is increasing and (R6) is satisfied we have $n_{k'} > 2n^\star$;
(3) for each $\tau' \in \Lambda^{k'}_{\tau_j}$ such that $B(m, \tau')$ holds there exist $\Lambda_{0,\tau'}$ and $\Lambda_{1,\tau'}$ which are $n_{k'}/2$-bushy above τ' on T_m and which are Ψ_{m+1}-splitting.

Choose l'' greater than:

$$\max\{|\Psi_{m+1}(\tau'')| : \tau'' \in \Lambda_{i,\tau'},\ i \leq 1,\ \tau' \in \Delta^{k'}_{\tau_j},\ j \leq k'\}.$$

By Lemma 6.1 we may then let $\Delta^{k'}_{\tau_{k'+1}}$ be a $g_m(l')/2^5$-bushy set of strings on T_m above $\tau_{k'+1}$ such that for all τ' in this set, $|\Psi_{m+1}(\tau')| > l''$. We then apply Lemma 6.2 to each pair $\Delta^{k'}_{\tau_j}, \Delta^{k'}_{\tau_{k'+1}}$ with $0 \leq j \leq k'$ (redefining $\Delta^{k'}_{\tau_{k'+1}}$ at each step in this process) in order to give sets $\Lambda^{k'+1}_{\tau_j}$ for each $0 \leq j \leq k'+1$ such that for each $j \neq j'$, $\Lambda^{k'+1}_{\tau_j}$ and $\Lambda^{k'+1}_{\tau_{j'}}$ are Ψ_{m+1}-splitting, and such that each $\Lambda^{k'+1}_{\tau_j}$ is $g_m(l')/(2^{5+3(k'+1)})$-bushy above τ_j. The fact that (R6) is satisfied suffices to show that at each step in this process the sets $\Delta^{k'}_{\tau_j}$ and $\Delta^{k'}_{\tau_{k'+1}}$ are sufficiently bushy that we can apply Lemma 6.2—the statement of Lemma 6.2 requires that n_0 and n_1 be positive multiples of 4, and since $(R3)'$ is satisfied $g_{m+1}(l) \geq 4$.

Once we have completed step k of this iteration, define $\Lambda_{\tau_j} = \Lambda^k_{\tau_j}$.

Step 2. We must ensure that T_{m+1} remains regular. In order to do so we use the same trick that we used when defining T_{m+1} to be Ψ_{m+1}-constant. For each τ of level l in T_{m+1} for which we have defined a value Λ_τ this is a (more than) $2g_{m+1}(l)$-bushy set of strings above τ. Let $l^\star$ be longer than any string in any Λ_τ. For each τ such that $\Lambda_\tau \downarrow$ search until either it is found that $B(m,\tau)$ does not hold or until we find Λ'_τ such that each string in this set is an extension of a string in Λ_τ, which is a $g_{m+1}(l)$-bushy set of strings on T_m above τ and such that every string in this set is of length $l^\star$. For each string τ of level l in T_{m+1} for which Λ'_τ is undefined, we decide that τ has no successors in T_{m+1}. For each string $\tau = T_{m+1}(\sigma)$ of level l in T_{m+1} for which Λ'_τ is defined let $\Lambda'_\tau = \{\tau_0, .., \tau_k\}$ and for all $k' \leq k$ define $T_{m+1}(\sigma \star k') = \tau_{k'}$, leaving $T_{m+1}(\sigma \star k')$ undefined for all $k' > k$.

It is clear that T_{m+1} defined in this way is delayed Ψ_{m+1}-splitting, partial computable and regular with computable domain. $T_{m+1}(\lambda)$ is DNR and is not terminal in T_{m+1}. If τ is terminal in T_{m+1} then $B(m,\tau)$ does not hold. We may therefore define $w_{m+1} = n^\star$.

7. Debts paid

It remains to specify how T_0 should be defined, how g_0 and w_0 should be defined, and how $g_{m+1}(l)$ should be defined for all $l > 0$ and $m \geq 0$. We define these values as follows. We define $g_0(0) = 2$ and $w_0 = 1$. For $l > 0$ and $m \geq 0$ we define $g_m(l) = (2^{5+3w(T_m,l)})^l$. We define $T_0(\lambda) = \lambda$. For each σ of length $l \geq 0$ such that $T_0(\sigma) \downarrow$ we define $T_0(\sigma \star k') = \sigma \star k'$ for each $k' < g_0(l)$ and for all $k' \geq g_0(l)$ we decide that $T_0(\sigma \star k')$ is undefined.

The only remaining tasks of any difficulty at all are to show that (R1) and (R6) are always satisfied. For any m and any $l \geq 1$ the fact that $g_m(l+1) > g_m(l)$ follows from the fact that $w(T_m, l')$ is nondecreasing as a function of l'. The fact that $g_m(1) > g_m(0)$ follows from the fact that $w(T_m, 1) \geq g_m(0)$. In order to see that (R6) is satisfied, observe first that for any l and m, $\kappa_{m+1}(l) > l$ and $w(T_{m+1}, l) \leq w(T_m, \kappa_{m+1}(l))$. Then $g_m(\kappa_{m+1}(l)) = (2^{5+3w(T_m,\kappa_{m+1}(l))})^{\kappa_{m+1}(l)} \geq (2^{5+3w(T_{m+1},l)})^{\kappa_{m+1}(l)} \geq (2^{5+3w(T_{m+1},l)})g_{m+1}(l)$.

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