EMPTY INTERVALS IN THE ENUMERATION DEGREES

THOMAS F. KENT, ANDREW E. M. LEWIS, AND ANDREA SORBI

Abstract. We construct a Π^0_2 enumeration degree which is a strong minimal cover.

1. Introduction

Enumeration reducibility is a natural variant of Turing reducibility, and may be considered a way of formalizing computation relative to partial information. Intuitively, a set of natural numbers B is enumeration reducible to A ($B \leq_e A$), if there is an algorithm which will enumerate the elements of B given any enumeration of A. This reducibility then gives rise to a degree structure in the usual way, so that we may consider the *e-degree* (enumeration degree) of any given set of natural numbers. For an introduction to research in this area we refer the reader to [2].

Of course a basic question regarding any degree structure is whether or not the structure is dense, and in [3], Cooper provided a negative answer for the e-degrees. Having shown [4] that the e-degrees of Σ^0_2 sets *are* dense, he then asked; what is the least n such that density does not hold for the e-degrees below $\mathbf{0}^{(n)}$? This was answered by Slaman and Calhoun in [1], who showed non-density for the e-degrees below $\mathbf{0}^{(2)}$. These results are established by constructing degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{b}$ is a *minimal cover* of $\boldsymbol{a}$, i.e. $\boldsymbol{b} > \boldsymbol{a}$ and there does not exist $\boldsymbol{c}$ with $\boldsymbol{a} < \boldsymbol{c} < \boldsymbol{b}$. Another very natural question along these lines, and one that has played an important role in the study of other degree structures, is as to which degrees, if any, have a *strong minimal cover*:

Definition 1.1. *We say $\boldsymbol{b}$ is a* strong minimal cover *for $\boldsymbol{a}$ if the degrees strictly below $\boldsymbol{b}$ are precisely those below and including $\boldsymbol{a}$. (Note that this implies $\boldsymbol{b} > \boldsymbol{a}$.)*

In $\mathcal{D}_m$ – the structure of the many-one degrees, induced by a strengthening of the Turing reducibility – Lachlan's proof that every m-degree has a strong minimal cover played a vital role in Ershov's [5] and Paliutin's [7] results characterizing the structure and in showing, for instance, that $\mathbf{0}_m$ is the only definable singleton. The question of characterizing those Turing degrees with a strong minimal cover was raised by Spector in his 1956 paper [9], and the question as to whether every minimal Turing degree has a strong minimal cover has remained one of the longstanding questions of degree theory. For a survey of research in this area, we refer the reader to [6]. The result of this paper is that there exists an e-degree which is a strong minimal cover:

Theorem 1.2. *There exists a (Π^0_2) e-degree which is a strong minimal cover.*

Notation and terminology will be standard unless explicitly stated otherwise, and will generally follow [8]. In particular, given strings α and β, $\alpha \subseteq \beta$ ($\alpha \subset \beta$) denotes that β extends (properly extends) α. We say α is to the left of β ($\alpha <_L \beta$) if α is lexicographically less than β but $\alpha \not\subseteq \beta$. Furthermore, by $\alpha \leq \beta$ we denote non-strict lexicographical ordering

The first author was partially supported by a Marie Curie Incoming International Fellowship of the European Community FP6 Program under contract number MIFI-CT-2006-021702. The second author was partially supported by a Marie-Curie Fellowship of the European Community Sixth Framework programme under contract number MEIF-CT-2005-023657 and by a Royal Society University Research Fellowship.

($\alpha <_L \beta$ or $\alpha \subseteq \beta$), and by $\alpha < \beta$ we denote strict lexicographical ordering ($\alpha \leq \beta$ and $\alpha \neq \beta$).

2. Proving the theorem

In order to prove the theorem we build sets A and B which meet the following requirements for all enumeration operators Ψ and Φ:

$$\begin{array}{ll} \mathcal{R} & : \exists\Lambda(B = \Lambda^A), \\ \mathcal{S}_\Psi & : \exists\Gamma(\Psi^A = \Gamma^B) \text{ or } \exists\Delta(A = \Delta^{\Psi^A}), \text{ and} \\ \mathcal{T}_\Phi & : A \neq \Phi^B, \end{array}$$

where Λ is an enumeration-operator which we build during the course of the construction. Both Γ and Δ are enumeration operators which are built by the construction but, unlike Λ, they are local to the strategy in which they are built.

The construction will be a tree argument by full approximation – no oracle will be used to build A, and B will simply be Λ^A. At the beginning of each stage of the construction we start with an approximation to A which is empty, and various numbers will then be enumerated into A during the stage. Ultimately we define $z \in A$ if there are infinitely many stages of the construction at which z is enumerated into A.

2.1. Informal description of the strategies.

The $\mathcal{S}$-strategy in isolation. In isolation, an $\mathcal{S}$-strategy α builds an operator Γ such that $\Psi^A = \Gamma^B$. When an axiom (x, F) is enumerated into Ψ it chooses a *coding location* $b_{x,F}$ and also an *axiom location* a (which are both specific to α and this axiom), then it enumerates the axiom $(x, \{b_{x,F}\})$ into Γ and the axiom $(b_{x,F}, \{a\})$ into Λ. It enumerates the triple (a, x, F) into a set $\mathcal{A}(\alpha)$. This set $\mathcal{A}(\alpha)$ is initially empty and is enumerated by the strategy. When any triple is enumerated into $\mathcal{A}(\alpha)$ it remains there until the strategy is initialized. At future stages s, the fact that the triple (a, x, F) is in $\mathcal{A}(\alpha)$, means that when $F \subseteq A_s$, a is enumerated into A.

Note that, in order that this procedure should suffice to ensure $\Gamma^B = \Psi^A$, we need that when $F \subseteq A$, there are infinitely many stages at which all elements of F are enumerated into A.

The interaction of $\mathcal{S}$-strategies. The only real interaction between $\mathcal{S}$ strategies is that they may cause each other to enumerate elements into A. Enumerations into A because of triples in $\mathcal{A}(\alpha)$ for any $\mathcal{S}$ strategy α, will not just be made by α, but by any strategy on the tree where α is active. Let α_0 and α_1 be $\mathcal{S}$-strategies, and let α_1 be of lower priority than α_0. When α_1 enumerates an element into A_s, this may cause $F \subset A_s$ for some F such that there is a triple $(a, x, F) \in \mathcal{A}(\alpha_0)$. Then α_1 will enumerate a into A_s. This may, in turn, cause $F' \subseteq A_s$ for some F' such that there is a triple $(a', x', F') \in \mathcal{A}(\alpha_1)$, so that α_1 then enumerates a' into A_s, and so on. At each stage the sets $\mathcal{A}(\alpha_0)$ and $\mathcal{A}(\alpha_1)$ will be finite.

The $\mathcal{T}$-strategy in isolation. In isolation, the $\mathcal{T}$-strategy acts like a modified Friedberg-Muchnik strategy. It continues to enumerate a witness z_0 into A at every stage, until z_0 enters Φ^B. Then it restrains B, and no longer enumerates z_0 into A.

One $\mathcal{T}$-strategy and one $\mathcal{S}$-strategy. This is the most interesting interaction of the construction. We shall describe not just how two such strategies might act in total isolation, but also, to some extent, the kind of environment that they provide for lower priority strategies. Let α be a $\mathcal{T}$ strategy which has lower priority than β which is an $\mathcal{S}$ strategy. The strategy α begins as normal and picks a witness z_0. It then enumerates z_0 into A every stage at which it acts, and allows β to enumerate new elements into A as described above in order to build Γ^B. If z_0 never enters Φ^B, then the $\mathcal{T}$ requirement is satisfied, so assume that at some stage s_0, we see $z_0 \in \Phi^B$. At this point α will cease to enumerate z_0 into A. The potential problem now is that this may cause β not to enumerate certain elements into A and thereby cause z_0 to leave Φ^B. Let ψ be the finite set Ψ^A as defined when z_0 enters Φ^B. The strategy α then waits for a stage $s_1 > s_0$ at which $z_0 \notin A$ and $\psi \subset \Psi^A$. If there exists such a stage s_1, then, without injuring β, we can now enumerate at every subsequent stage all those elements which β stopped enumerating into A when α stopped enumerating z_0 (of course z_0 itself will not be an axiom location for β, so we can do this while keeping z_0 out of A). In this case, then, we satisfy the $\mathcal{T}$ requirement. So long as there does not exist such a stage s_1, however, α passes z_0 on to certain lower priority requirements to work with. These strategies may cause $z_0 \in A$ or otherwise, but so long as the stage s_1 does not exist, we have $z_0 \in A$ iff $\psi \subseteq \Psi^A$ – we have successfully completed our first step in building Δ. While waiting for the stage s_1, α then continues with a new witness. If every witness it chooses fails to provide a diagonalization, then α provides this sequence of witnesses as an infinite *stream* of numbers for lower priority strategies to work with (those lower priority strategies which correctly guess this outcome for α), and produces an enumeration operator Δ which guarantees that the intersection of A with this stream is enumeration reducible to Ψ^A. It is not difficult to prove then that, in fact, A as a whole is enumeration reducible to Ψ^A. In this case, therefore, the $\mathcal{T}$-requirement is not yet satisfied, but we have satisfied the $\mathcal{S}$ requirement by showing $A \leq_e \Psi^A$.

We may consider, then, a version of the strategy α which proceeds roughly as follows. This version of the strategy does not take into account, among other things, that strategies of priority less than α, which guess that α successfully diagonalizes, will also have to have numbers available to choose as axiom locations and witnesses.

(1) Choose a fresh witness z_0.
(2) Wait until a stage s_0 at which $z_0 \in \Phi^{\Lambda^{A \cup g(z_0)}}$, where A is the set of numbers already enumerated into A during the course of the stage before α is made eligible to act, and $g(z_0)$ is z_0 together with the set that β will require enumerated into A if z_0 is enumerated in. Meanwhile, enumerate $g(z_0)$ into A at each stage.
(3) When such a stage s_0 is found do the following:
  (a) Define ψ^{z_0} to be the finite set $\Psi^{A \cup g(z_0)}$;
  (b) z_0 is now call a *realized* witness;
  (c) Enumerate the axiom (z_0, ψ^{z_0}) into Δ;
  (d) Enumerate z_0 into S (the stream produced by α);
  (e) Return to Step (1).

$*$ If ever we see $\psi^z \subseteq \Psi^{A \cup g(S-\{z\})}$ for some realized witness z when performing step (3) at some stage s_1, then keep z out of A permanently and stop. At all future stages, enumerate into A every number in the value $A \cup g(S - \{z\})$ which we saw at stage s_1 (to ensure $\psi^z \subset \Psi^A$), together with all numbers in $\{A \cup g(z)\} - \{z\}$ (to ensure $z \in \Phi^B$), as this value was defined when z became a realized witness. In the actual construction the latter task will be achieved through the use of 'doubling' requests.

2.2. **Dealing with all requirements.** In order to satisfy all requirements there are various modifications which have to be made to the procedures described above. To describe these changes in detail involves formally defining the construction. Before doing so, however, we make some preliminary comments which will aid understanding of the following subsections.

- Each $\mathcal{T}$ strategy α, may now have a number of $\mathcal{S}$ strategies $\beta \subset \alpha$ and which are active at α. This is not difficult to deal with, the strategy now builds a functional Δ_i corresponding to each, one of which will be successful if α fails to diagonalize. This

Δ_i will specify an enumeration of the corresponding stream from the relevant Ψ^A–from this it will be easy to deduce that A as whole is enumeration reducible to Ψ^A.

- The $\mathcal{T}$ strategy described above did not take into account that strategies of priority below α, which guess that α successfully diagonalizes, will also need to define axiom locations and witnesses. These strategies will work by choosing (different) numbers from the same stream as α. The set of numbers that these strategies enumerate into A can also be instrumental in causing z_0 to enter Φ^B, where z_0 is the witness for α. Once this happens, then, α enumerates all of these numbers into a set U. Strategies for which numbers in U were previously axiom locations or witness will be initialized and α will enumerate all numbers in U into A at every subsequent stage at which it acts. This is done so as to ensure the correctness of Δ – we want that when $z_0 \in A$ we are guaranteed $\psi^{z_0} \subseteq \Psi^A$. For each α, the set U, which is specific to α (we may write as U_α when we wish to clarify the situation), is initially empty. When numbers are enumerated into this set, they do not leave until α is initialized. The set U can be seen (intuitively) as 'junk' which α has collected since it was last initialized, and which it will now enumerate into A every stage at which it acts.
- When the *doubling request* (b, x, F) is made of an $\mathcal{S}$ strategy α, this requires α to choose some new axiom location a, enumerate a triple (a, x, F) into $\mathcal{A}(\alpha)$, and an axiom $(b, \{a\})$ into Λ. The doubling request, in other words, causes us to choose another element a such that $a \in A$ means $b \in B$.

2.3. **The tree of strategies.** Fix an arbitrary effective priority ordering $\{R_e\}_{e\in\omega}$ of all $\mathcal{S}$- and $\mathcal{T}$-requirements. The $\mathcal{R}$-requirement will not be put on the tree of strategies as it is handled globally.

Let $\Sigma = \{\text{stop} < \infty_0 < \infty_1 < \infty_2 < \cdots < \text{wait}\}$ be our set of outcomes. We define $\mathbf{T} \subset \Sigma^{<\omega}$ and refer to it as our *tree of strategies*. Each node of $\mathbf{T}$ will be associated with (and, in fact, identified with) a strategy.

We assign requirements to nodes on $\mathbf{T}$ by induction as follows. The empty node is assigned requirement R_0, and no requirement is labelled *active* or *satisfied along* the empty node. Given an assignment to a node $\alpha \in \mathbf{T}$, we distinguish cases depending on the requirement R assigned to α:

Case 1: R is an $\mathcal{S}$-requirement. Then label R active along $\alpha^\frown\langle\text{wait}\rangle$ via α. For all other requirements R', label R' active or satisfied along $\alpha^\frown\langle\text{wait}\rangle$ via $\beta \subset \alpha$ if and only if it is so along α. Assign to $\alpha^\frown\langle\text{wait}\rangle$ the highest priority requirement that is labelled neither active nor satisfied along $\alpha^\frown\langle\text{wait}\rangle$.

Case 2: R is a $\mathcal{T}$-requirement. Let $\beta_0 \subset \cdots \subset \beta_{i_0-1}$ be all the strategies $\beta \subset \alpha$ such that some $\mathcal{S}$-requirement is active along α via β_i. We denote by $\mathcal{S}_i$ the $\mathcal{S}$-requirement for β_i. (Here we allow $i_0 = 0$.) For $o \in \{\text{stop}, \text{wait}\}$, label R satisfied along $\alpha^\frown\langle o\rangle$ via α; and for all other requirements R', label R' active or satisfied along $\alpha^\frown\langle o\rangle$ via $\beta \subset \alpha$ if and only if it is so along α. If $i_0 > 0$, fix $i \in [0, i_0)$. Label $\mathcal{S}_i$ satisfied along $\alpha^\frown\langle\infty_i\rangle$ via β_i and label any $\mathcal{S}_j$ requirement, for $j \in [0, i_0) - \{i\}$, active along $\alpha^\frown\langle\infty_i\rangle$; any other requirement is labelled active or satisfied along $\alpha^\frown\langle\infty_i\rangle$ via $\beta \subset \alpha$ if and only if it is so along α. For any outcome $o \in \{\text{stop}, \text{wait}\} \cup \{\infty_i : i \in [0, i_0)\}$, assign to $\alpha^\frown\langle o\rangle$ the highest priority requirement labelled neither active nor satisfied along $\alpha^\frown\langle o\rangle$. (The intuition is that under the finitary outcomes in $\{\text{stop}, \text{wait}\}$, the $\mathcal{T}$-requirement is assumed to be satisfied finitarily by diagonalization; whereas under outcome ∞_i, the $\mathcal{S}_i$-requirement is now assumed to be satisfied by α constructing an enumeration operator Δ_i, while all other $\mathcal{S}_j$-requirements active along α via some $\beta_j \subset \alpha$ are in fact uninjured.)

The *tree of strategies* $\mathbf{T}$ is now the set of all nodes $\alpha \in \Sigma^{<\omega}$ to which requirements have been assigned.

2.4. **The construction.** We give first some conventions and definitions. The construction proceeds in stages $s \in \omega$. Each stage $s > 0$ is composed of substages $t < s$ such that some strategy $\alpha \in \mathbf{T}$, with $|\alpha| = t$, acts at substage t of stage s and decides which strategy β will be *eligible to act* at substage $t+1$. If β is made eligible to act, then it acts so long as $|\beta| < s$. At substage 0 of stage $s > 0$, the root node $\emptyset$ is eligible to act. The longest strategy eligible to act during a stage s is called the current approximation to the true path at stage s and is denoted f_s.

At each stage s we define a set $A_s = \bigcup_{t<s} A_{s,t}$ where $A_{s,t}$ is the set of all elements which are enumerated into A during stage s at a substage strictly less than t. We drop the subscripts s and t when their values are clear.

During the construction, all parameters are assumed to remain unchanged unless specified otherwise. If ever we need to refer to a functional or a parameter of a strategy β, we may clarify the relationship by marking it either with a subscript, as in Φ_β, or with a superscript, as in z_0^β. We assume each given functional to contain only a finite number of axioms at each stage of its enumeration.

For each $\alpha \in \mathbf{T}$, we will define a stream of numbers $S(\alpha)$. The stream $S(\emptyset)$ of the root node $\emptyset$ is defined to be $[0, s)$ at any stage s. The stream $S(\alpha^\frown\langle o\rangle)$ is just $S(\alpha)$, unless $o = \infty_i$ for some i. We shall explicitly enumerate each $S(\alpha^\frown\langle\infty_i\rangle)$ during the course of the construction. At any point during the construction, we let $S^*(\alpha)$ denote the set of numbers enumerated into $S(\alpha)$ and which are not in $U_{\alpha'}$, and are not an axiom location, a witness or realized witness for α', for any strategy α' of strictly higher priority than α such that $S(\alpha') = S(\alpha)$. Suppose α is the ith string in $\mathbf{T}$, according to some fixed computable enumeration of this tree. At any point during stage s, an element z of $S(\alpha)$ is *suitable for* α, if it is not in $U_{\alpha'}$ for any α' of higher priority than α, is not already an axiom location or a witness or realized witness for α, $z \geq i$, z is greater than the last stage at which α was initialized, and is the $\langle i, j\rangle$th number enumerated into $S(\alpha)$ for some j.

When we *initialize* a strategy α, we undefine all local parameters, set all local functionals equal to the empty set, put $\mathcal{A}(\alpha) = \emptyset$, $U_\alpha = \emptyset$ and if α is of the form $\beta^\frown\langle\infty_i\rangle$ for some i and β then we set $S(\alpha) = \emptyset$. We now regard the set of realized witnesses, witnesses, axiom locations and coding locations for this strategy as empty.

At each stage s the construction proceeds as follows.

Stage 0*:* Initialize all $\alpha \in \mathbf{T}$.
Substage t of stage $s > 0$: Let α be eligible to act. Let $\beta_0 \subset \cdots \subset \beta_{i_0-1}$ be all of the strategies such that some $\mathcal{S}_i$ is active along α via β_i (allowing $i_0 = 0$).

Definition 2.1. *Let $X \subseteq \omega$. Define $g_\alpha(X) = \bigcup_{n<\omega} X_n$, where $X_0 = X \cup A$ and for each $n \geq 0$,*

$$X_{n+1} = X_n \cup \{a : i < i_0, (a, x, F) \in \mathcal{A}(\beta_i), x \in \omega, F \subseteq X_n\}$$

Roughly speaking, then, the function g_α tells us what A will look like should we enumerate in all numbers in X, once all the $\mathcal{S}$ strategies β_i have acted to deal with these enumerations. At any given stage the sets $\mathcal{A}(\beta_i)$ will be finite.

We now distinguish the action we take depending on the requirement R assigned to α.

Case 1. R is an $\mathcal{S}_\Psi$-requirement.

(1) Check to see whether a, which is the number most recently enumerated into $S(\alpha)$, is suitable for α. If not proceed to (2). Otherwise, there are two subcases to consider:

Case 1.1. Of all axioms enumerated into Ψ and all doubling requests made of α, and which are yet to receive an axiom location, the first enumerated is an axiom (x, F) in Ψ. If a is greater than all elements of F and was enumerated into $S(\alpha)$ after (x, F)

was enumerated into Ψ then choose a coding location $b_{x,F}$ larger than any number seen so far in the construction, enumerate $(x, \{b_{x,F}\})$ into Γ, enumerate (a, x, F) into $\mathcal{A}(\alpha)$ and enumerate the axiom $(b_{x,F}, \{a\})$ into Λ. In this case we call a the axiom location corresponding to the axiom (x, F) in Ψ. In either case proceed to (2).

Case 1.2. Otherwise. Then let (b, x, F') be the first enumerated doubling request made of α which is yet to receive an axiom location, or if there exists no such then go straight to (2). If a is greater than all elements of F' and was enumerated into $S(\alpha)$ after the doubling request (b, x, F') was made of α, enumerate (a, x, F') into $\mathcal{A}(\alpha)$, and enumerate the axiom $(b, \{a\})$ into Λ. In this case we call a the axiom location corresponding to the doubling request (b, x, F'). In either case proceed to (2).

(2) Enumerate $g_\alpha(\emptyset)$ into A and end the current substage by letting $\alpha^\frown\langle\text{wait}\rangle$ be eligible to act.

Case 2. R is a $\mathcal{T}_\Phi$-requirement. Recall that $\beta_0 \subset \cdots \subset \beta_{i_0-1}$ are the strategies such that some $\mathcal{S}_i$ is active along α via β_i. For $i \in [0, i_0)$, the enumeration operators Ψ_i and Γ_i are those of β_i. Pick the first case which applies.

Case 2.1. The witness z_0 is undefined. Define z_0 to be the number most recently enumerated into $S(\alpha)$ if this number is suitable for α and was enumerated into $S(\alpha)$ after the last stage at which the witness z_0 was made undefined. Enumerate $g_\alpha(U)$ into A. End the substage by letting $S(\alpha^\frown\langle\text{wait}\rangle)$ be eligible to act.

Case 2.2. Otherwise, z_0 is defined. Choose the first subcase which applies.

Case 2.2.1. α has been declared successful (and has not been initialized since then). Enumerate $g_\alpha(U)$ into A and end the substage by letting $\alpha^\frown\langle\text{stop}\rangle$ be eligible to act next.

Case 2.2.2. Letting $V = g_\alpha(S^*(\alpha^\frown\langle\text{wait}\rangle) \cup U \cup \{z_0\})$, $z_0 \notin \Phi^{\Lambda^V}$. This means that if we let $\alpha^\frown\langle\text{wait}\rangle$ be eligible to act then, no matter how many numbers we enumerate into A during subsequent substages of stage s, z_0 will not be in Φ^{Λ^A} at the end of the stage. End the substage by enumerating $g_\alpha(\{z_0\} \cup U)$ into A and letting $\alpha^\frown\langle\text{wait}\rangle$ be eligible to act next.

Case 2.2.3. Otherwise, $z_0 \in \Phi^{\Lambda^V}$. We now call z_0 a *realized* witness. Enumerate all elements of $S^*(\alpha^\frown\langle\text{wait}\rangle)$ into U. Define $W_{z_0} = \Lambda^V$ – if we can ensure that $W_{z_0} \subset B$, then we shall have that $z_0 \in \Phi^B$. For each $i < i_0$, define $\psi_i^{z_0} = \Psi_i^V$ (where we consider, of course, only axioms enumerated into Ψ by stage s, so that $\psi_i^{z_0}$ is a finite set of numbers). Enumerate z_0 into $S(\alpha^\frown\langle\infty_{i_0-1}\rangle)$. For each $i < i_0$ and each $z \in S(\alpha^\frown\langle\infty_i\rangle)$, set:

$$G_{i,z} = (U \cup \bigcup_{i \leq j < i_0} S(\alpha^\frown\langle\infty_j\rangle)) - \{z\}.$$

We now distinguish two subcases. In what follows, note that ψ_i^z for each $i < i_0$ and each realized witness z, is defined at the point at which z becomes a realized witness, and does not change at subsequent stages (unless initialization occurs). The value $G_{i,z}$, however, is dependent upon the stage considered.

Definition 2.2. *For $i \in [0, i_0)$, the realized witness z is* Γ_i-cleared *if $\psi_i^z \subseteq \Psi_i^{g_\alpha(G_{i,z})}$.*

Case 2.2.3.1. Some realized witness $z \notin U$ is Γ_i-cleared for all $i \in [0, i_0)$. Then α is *declared successful.* If $i_0 > 0$ enumerate all elements of $G_{0,z}$ into U. For each b which is a coding location $b_{x,F}$ for some β_i, $i < i_0$, and such that $b \in W_z - \Lambda^{g_\alpha(U)}$, make the doubling request $(b, x, g_\alpha(U))$ of β_i. Enumerate $g_\alpha(U)$ into A and let $\alpha^\frown\langle\text{stop}\rangle$ be eligible to act.

Case 2.2.3.2. Otherwise. Find the lexicographically least pair (i,z) (if any) such that $i < i_0$ and z is a realized witness such that $z \notin U$ and such that $z \notin S(\alpha^\frown\langle\infty_j\rangle)$ for all $j \leq i$, and z is Γ_j-cleared for all $j \in (i, i_0)$. If there exists no such pair then let $(i,z) = (i_0 - 1, z_0)$ and go to (3) below.

(1) Enumerate z into $S(\alpha^\frown\langle\infty_i\rangle)$;
(2) Enumerate $\bigcup_{i<j<i_0} S(\alpha^\frown\langle\infty_j\rangle) - \{z\}$ into U, empty each $S(\alpha^\frown\langle\infty_j\rangle)$ and each Δ_j such that $i < j < i_0$;
(3) Make the witness z_0 undefined;
(4) Enumerate the axiom (z, ψ_i^z) into Δ_i;
(5) Enumerate $g_\alpha(U)$ into A;
(6) End the current substage by allowing $\alpha^\frown\langle\infty_i\rangle$ to be eligible to act next.

Ending the stage s: Initialize all $\alpha >_L f_s$.

2.5. **The verification.** Let f be the *true path* of the construction, defined as:

$$f(n) = \liminf_{\{s>n\}} f_s(n)$$

where lim inf is taken with respect to the lexicographical ordering on $\mathbf{T}$.

Definition 2.3. *Define* $A = \{x : \forall s\, \exists t > s\, (x \in A_t)\}$, *and set* $B = \Lambda^A$.

Note that A is Π^0_2 by definition. The following facts are easily derived:

I Each $\alpha \subset f$ is initialized only finitely often.
II For each strategy $\alpha \subset f$ the stream $S(\alpha)$, which is enumerated subsequent to its final initialization, is infinite.
III If (a, x, F) in $\mathcal{A}(\alpha)$ then $a > n$ for every $n \in F$.
IV If α acts at stages $s_1 < s_2$, has the same outcome o at these stages and $\alpha^\frown\langle o\rangle$ is not initialized at any intermediate stage, then the set of numbers α enumerates into A at stage s_2 is a superset (not necessarily proper) of the set of numbers it enumerates into A at stage s_1.
V For each z, at the end of stage s, there is at most one α such that either:
  (a) $z \in U_\alpha$;
  (b) z is a witness (but not a realized witness) for α;
  (c) z is an axiom location for α.
  For such α, exactly one of (a),(b) or (c) holds.
VI For any $\mathcal{S}$-requirement R_e, either there is a unique strategy $\alpha \subset f$ such that R_e is assigned to α, and in this case R_e is active along all β with $\alpha \subset \beta \subset f$, or else there is a shortest $\gamma \subset f$ such that R_e is labelled satisfied along all β with $\gamma \subset \beta \subset f$. For any $\mathcal{T}$-strategy R_e, there is a longest $\alpha \subset f$ to which R_e is assigned. R_e is labelled satisfied along all β with $\alpha \subset \beta \subset f$.

We say z has *final allocation* (α, ρ), if $\rho \in \{\mathrm{ax}, \mathrm{wit}, U, \mathrm{r}\}$ and either:

a. $\rho = \mathrm{ax}$ and there is a point of the construction after which z is always an axiom location for α;
b. $\rho = \mathrm{wit}$ and there is a point of the construction after which z is always a (non-realized) witness for α;
c. $\rho = U$ and there is a point of the construction after which z is always in U_α;
d. $\rho = \mathrm{r}$ and there is a point of the construction after which z is always a realized witness for α, is not a realized witness for any α' unless $\alpha' \subseteq \alpha$, and is not a (non-realized) witness, axiom location, or in $U_{\alpha'}$, for any α'.

Lemma 2.4. *Every* $z \in \omega$ *has a final allocation.*

Proof. It follows from the fact that there infinitely many Φ such that $\Phi^C = \omega$ for all C, that any z becomes a witness or an axiom location, or is enumerated into $U(\alpha)$, for some strategy α at some point during the construction. This follows, since then some witness for a strategy α with $S(\alpha) = S(\emptyset)$ becomes realized at a stage $s > z$, and then z will be enumerated into $U(\alpha)$ if it is not already a witness, a realized witness or an axiom location, or in $U_{\alpha'}$ for some strategy α'.

Next observe that, when z is selected as a witness or an axiom location for α we have that $z > i$, where α is the ith string in $\mathbf{T}$, and z is also greater than the last stage at which α was initialized. Therefore z can only be chosen as a witness or an axiom location at a finite number of stages of the construction. If z is a witness, an axiom location or in U_α for some strategy α and, for some $\alpha' \not\subseteq \alpha$, z is subsequently an axiom location or a witness for α', or in $U_{\alpha'}$, then in the interim β which is the longest string compatible with both α and α', must have z as a realized witness and must have enumerated z into $S(\beta^\frown\langle\infty_i\rangle)$ for some i such that $\beta^\frown\langle\infty_i\rangle \subseteq \alpha$. Clearly this can only take place a finite number of times, since any given z can only be a realized witness for a finite number of strategies and each such strategy only has a finite number of outcomes. The result then follows from the fact if z is a realized witness for both α and β at the end of any stage, α and β are compatible. □

Lemma 2.5. *If $z \in A$ then there exists $\alpha \subset f$ which enumerates z into A at all but finitely many of the stages at which it acts.*

Proof. We prove the statement of the lemma by induction on z. So suppose the result holds for all $z' < z$ and let (α, ρ) be the final allocation of z. If $\rho \in \{\mathrm{wit}, \mathrm{U}\}$, then either $\alpha \not\subset f$, in which case $z \notin A$, or else α is a strategy on the true path which enumerates z into A at all but finitely many stages at which it acts. If $\rho = \mathrm{r}$, then $z \notin A$. Finally, suppose that $\rho = \mathrm{ax}$. If $\alpha \not\subset f$, then $z \notin A$, so suppose otherwise. Then there exists a unique pair x, F such that $(a, x, F) \in \mathcal{A}(\alpha)$. By III, observed previously, all elements of F are less than a. If $F \not\subseteq A$ then $z \notin A$, so suppose otherwise. By the induction hypothesis, there exists $\beta \supseteq \alpha$ on the true path such that all elements of F have already been enumerated into A at any stage and substage at which β acts. Then β will enumerate z into A at all but finitely many stages at which it acts since $z \in g_\beta(\emptyset)$ at all but finitely many of such stages.

□

Lemma 2.6. *Each $\mathcal{T}$ requirement is satisfied.*

Proof. Let R be a $\mathcal{T}$ requirement. By VI there is a longest $\alpha \subset f$ to which R is assigned, and R is labelled satisfied along all β with $\alpha \subset \beta \subset f$.

We consider first the case that $\alpha^\frown\langle\mathrm{wait}\rangle \subset f$. By I and II there is a stage of the construction s_0, subsequent to the final stage at which α is initialized, at which α chooses a witness z_0 which never becomes realized. It suffices to show that $z_0 \notin \Phi^B$, so suppose otherwise. Then there exists a finite set $F \subseteq A$ such that $z_0 \in \Phi^{\Lambda^F}$, and a finite number of axioms in Λ and Φ which ensure this is the case. By lemma 2.5, we can choose $s_1 > s_0$ greater than the stage at which any of these axioms are enumerated, such that for all $x \in F$ there exists $\alpha_x \subset f$ which enumerates x into A every stage $> s_1$ at which it acts. Let $s_2 > s_1$ be a stage at which all the α_x and α act. By induction on the length of α', it follows that the set of numbers enumerated into A by any $\alpha' \supset \alpha$ at stage s_2, is a subset of $g_\alpha(S^*(\alpha^\frown\langle\mathrm{wait}\rangle) \cup U \cup \{z_0\})$ (as this value is defined when α acts at stage s_2). Therefore z_0 would become a realized witness at stage s_2, which gives us the required contradiction.

Next we consider the case that $\alpha^\frown\langle\mathrm{stop}\rangle \subset f$. As in the instructions for α, let $\beta_0 \subset \cdots \subset \beta_{i_0-1}$ be all of the strategies such that some $\mathcal{S}_i$ is active along α via β_i, and for $i \in [0, i_0)$, let the enumeration operator Γ_i be that of β_i. Then at some stage s after which it is never initialized, α acts and is declared successful because some realized witness $z \notin U$ is Γ_i-cleared for all $i \in [0, i_0)$. Let $g_\alpha(U)$ be as defined at the end of stage s. From the definition of g_α and from IV it follows that any element b of $W_z - \Lambda^{g_\alpha(U)}$ is a coding location $b_{x,F}$ for some β_i, $i < i_0$. Since $g_\alpha(U)$ is a subset of (the ultimate value) A and α enumerates the doubling request $(b, x, g_\alpha(U))$, it follows that $b \in B$. Thus $z \in \Phi^B$ and $z \notin A$. □

Lemma 2.7. *Each $\mathcal{S}$ requirement is satisfied.*

Proof. Let R be a $\mathcal{S}$ requirement. By VI it suffices to divide into two cases.

We consider first the case that R is assigned to α which is active along all $\beta \supset \alpha$ such that $\beta \subset f$. In this case we wish to show that $\Psi(A) = \Gamma(B)$. Suppose $x \in \Psi(A)$. Then there is $(x, F) \in \Psi$ such that $F \subset A$, and α chooses an axiom location a and enumerates (a, x, F) into $\mathcal{A}_\alpha$, an axiom $(b, \{a\})$ into Λ, and an axiom $(x, \{b\})$ into Γ. It follows from Lemma 2.5 that $a \in A$ and therefore $x \in \Gamma^B$. Now suppose that $x \in \Gamma^B$. Then for some coding location b, an axiom $(x, \{b\})$ is enumerated into Γ, and an axiom $(b, \{a\})$ is enumerated into Λ such that $a \in A$. Since $a \in A$ it follows that there exists $(a, x, F) \in \mathcal{A}(\alpha)$ with $F \subset A$. Whether this triple is enumerated into $\mathcal{A}(\alpha)$ while following the instructions for case 1.1 or case 1.2 of the instructions for R, it follows that $x \in \Psi^A$ since a doubling request (b, x, F) is only enumerated if $x \in \Psi^F$.

Next we consider the case that there exists some shortest $\alpha^\frown\langle\infty_i\rangle \subset f$ such that R is labelled satisfied along all β with $\alpha^\frown\langle\infty_i\rangle \subseteq \beta$. Let $S(\alpha^\frown\langle\infty_i\rangle)$ take its final value i.e., the set enumerated subsequent to the final time that $\alpha^\frown\langle\infty_i\rangle$ is initialized. We show first that, for any $z \in S(\alpha^\frown\langle\infty_i\rangle)$, $z \in A$ iff $z \in \Delta_i(\Psi(A))$. Let ψ_i^z be as defined in the instructions for α. If $z \in A$ then it follows from IV that $\psi_i^z \subset \Psi^A$, since when z becomes a realized witness for α, we enumerate $S^*(\alpha^\frown\langle\text{wait}\rangle)$ into U_α. If $z \notin A$, then it suffices to show that $\psi_i^z \not\subset \Psi^A$, so suppose otherwise. Now we may argue almost exactly as in the proof of Lemma 2.6. There must exist a finite set $F \subseteq A$ such that $\psi_i^z \subseteq \Psi^F$, and a finite number of axioms in Ψ which ensure this is the case. By lemma 2.5, we can choose s_0 greater than the stage at which any of these axioms are enumerated, after which z is never enumerated into A, after which $\alpha^\frown\langle\infty_i\rangle$ is never initialized, and such that for all $x \in F$ there exists $\alpha_x \subset f$ which enumerates x into A every stage $> s_0$ at which it acts. Let $s_1 > s_0$ be a stage at which all the α_x and $\alpha^\frown\langle\infty_i\rangle$ act. By induction on the length of α', it follows that the set of numbers enumerated into A by any $\alpha' \supset \alpha$ at stage s_1, is a subset of $g_\alpha(G_{i,z})$ (as this value is defined when α acts at stage s_1). Therefore z would become Γ_i-cleared at stage s_1, which gives us the required contradiction.

Finally, we must deal with those z which lie outside $S(\alpha^\frown\langle\infty_i\rangle)$. Let $s_1 < s_2 < s_3 \cdots$ be those stages subsequent to the final stage at which $\alpha^\frown\langle\infty_i\rangle$ is initialized, at which α acts and has outcome $\alpha^\frown\langle\infty_i\rangle$. Let $g_{\alpha,k}$ be g_α as defined when α acts at stage s_k. Then $A = A^*$ which is the set c.e. in Ψ^A defined as:

$$A^* = \bigcup_k g_{\alpha,k}(A_{s_k,|\alpha|+1} \cup \Delta_i(\Psi^A)).$$

We prove by induction on z that $z \in A$ iff $z \in A^*$. Suppose the result holds for all $z' < z$. If $z \in A^*$ then it follows from IV and the fact that, for any $z' \in S(\alpha^\frown\langle\infty_i\rangle)$, $z' \in A$ iff $z' \in \Delta_i(\Psi(A))$, that $z \in A$. Suppose $z \in A$. By Lemma 2.5 there exists $\beta \subset f$ which enumerates z into A at all but finitely many stages at which it acts. If $\beta \subseteq \alpha$ or if $z \in S(\alpha^\frown\langle\infty_i\rangle)$ then it is clear that $z \in A^*$. Otherwise, z must be an axiom location for some $\mathcal{S}$ strategy β' which is active at α, and there exists a unique triple in $\mathcal{A}(\beta')$ of the form (z, x, F) such that $x \in \omega$ and $F \subset A$. Since every element of F is less than z, it follows by the induction hypothesis that all elements of F are in A^*, and therefore that $z \in A^*$.

□

References

[1] Calhoun, W.C. and Slaman, T.A., The Π^0_2 enumeration degrees are not dense, *Journal of Symbolic Logic*, 61, 1364-1379, 1996.

[2] Cooper, S.B., *Computability Theory*, Chapman & Hall, 2004.

[3] Cooper, S.B., Enumeration reducibility, non-deterministic computations and relative computability of partial functions, *Recursion theory week*, *Lecture Notes in Mathematics*, 1432, Springer-Verlag, Berlin, 57-110, 1989.

[4] Cooper, S.B., Partial degrees and the density problem, Part 2: the enumeration degrees of Σ^0_2 sets are dense, *Journal of Symbolic Logic*, 49, 503-513, 1984.

[5] Ershov, Y.L., The uppersemilattice of enumerations of a finite set, *Algebra and Logic*, 14, 258-284, 1975.

[6] Lewis, A.E.M., Strong minimal covers and a question of Yates: the story so far, *Proceedings of the Logic Colloquium 2006*.

[7] Paliutin, E., Addendum to the paper of Ershov, *Algebra and Logic*, 14, 284-287, 1975.

[8] Soare, R.I, *Recursively enumerable sets and degrees*, Springer-Verlag, Berlin, 1987.

[9] Spector, C., On degrees of recursive unsolvability, *Annals of Mathematics*, 64, 581-592, 1956.