Π_1^0 CLASSES, STRONG MINIMAL COVERS AND HYPERIMMUNE-FREE DEGREES

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ABSTRACT. We investigate issues surrounding an old question of Yates' as to the existence of a minimal degree with no strong minimal cover, specifically with respect to the hyperimmune-free degrees.

1. INTRODUCTION

If A and B are sets of natural numbers we write $A \leq_T B$ if A is Turing reducible to B—intuitively, A can be computed if we are able to compute B. The Turing reducibility induces an equivalence relation on the sets of natural numbers and an order $<$ on the equivalence classes. These equivalence classes are called the Turing degrees. Intuitively each degree represents an information content, since all sets in the same degree code the same information. An old question of Yates' remains one of the longstanding problems of degree theory:

Definition 1.1. *We write $\mathbf{0}$ to denote the least Turing degree. For any degree $\mathbf{a}$ we write $\mathcal{D}[< \mathbf{a}]$ in order to denote the set of degrees strictly below $\mathbf{a}$. A degree $\mathbf{a}$ is* ***minimal*** *if $\mathcal{D}[< \mathbf{a}] = \{\mathbf{0}\}$. A degree $\mathbf{b}$ is a* ***strong minimal cover*** *for $\mathbf{a}$ if $\mathcal{D}[< \mathbf{b}] = \mathcal{D}[\leq \mathbf{a}]$.*

Question 1.1 (Yates). *Does every minimal degree have a strong minimal cover?*

It seems fair to say that for a long time very little progress was made in the attempt to understand issues surrounding Yates' problem. This situation changed relatively recently, however, with Ishmukhametov's characterization of the computably enumerable (c.e.) degrees which have a strong minimal cover. The first ingredient here was provided Downey, Jockusch and Stob [DJS] in 1996. They defined a degree $\mathbf{a}$ to be **array nonrecursive** (or **array incomputable**) if for each $f \leq_{wtt} K$ there is a function g computable in $\mathbf{a}$ such that $g(n) \geq f(n)$ for infinitely many n, where K denotes Turing's halting problem, and $f \leq_{wtt} g$ if $f \leq_T g$ and there is a computable bound on the number of arguments of g which are required on any given input.

Theorem 1.1 (Downey,Jockusch,Stob [DJS]). *Given $\mathbf{a}$ which is a.i.c.:*
  (1) *$\mathbf{a}$ is not minimal,*
  (2) *if $\mathbf{c} > \mathbf{a}$ then there is a degree $\mathbf{b} < \mathbf{c}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$.*

Ishmukhametov then combined this work to great effect with an analysis of the **c.e. traceable** degrees (referred to by him as **weakly recursive**):

Definition 1.2. $A \subseteq \omega$ *is* ***c.e. traceable*** *if there is a computable function* p *such that for every function* $f \leq_T A$ *there is a computable function* h *such that* $|W_{h(n)}| \leq p(n)$ *and* $f(n) \in W_{h(n)}$ *for all* $n \in \omega$.

Having observed that the class of c.e. traceable degrees (those whose sets are c.e. traceable) is complementary to the class of a.i.c. degrees in the c.e. degrees, Ishmukhametov [SI] was then able to show that all c.e. traceable degrees have a strong minimal cover.

Theorem 1.2 (Ishmukhametov [SI])**.** *A c.e. degree has a strong minimal cover iff it is c.e. traceable.*

Definition 1.3. *We say a degree* $\mathbf{a}$ *satisfies the* ***cupping property*** *if for every* $\mathbf{c} > \mathbf{a}$ *there exists* $\mathbf{b} < \mathbf{c}$ *with* $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$.

It is worth remarking that, as an immediate consequence of these results, every c.e. degree either has a strong minimal cover or satisfies the cupping property. It is an open question as to whether this is true of the Turing degrees in general.

Definition 1.4. *Given any* $T \subseteq 2^{<\omega}$ *and* $\tau, \tau' \in T$, *we say that* τ *is a* ***leaf*** *of* T *if it has no proper extensions in* T. *When* $\tau \subset \tau'$ *we call* τ' *a* ***successor*** *of* τ *in* T *if there doesn't exist* $\tau'' \in T$ *with* $\tau \subset \tau'' \subset \tau'$.

Definition 1.5. *Given any* $T \subseteq 2^{<\omega}$ *and* $A \subseteq \omega$ *we denote* $A \in [T]$ *if there exist infinitely many initial segments of* A *in* T.

The following definition is slightly non-standard, but seems convenient where the discussion of splitting trees with unbounded branching is concerned:

Definition 1.6. *We say that* $T \subseteq 2^{<\omega}$ *is a* ***c.e. tree*** *if it has a computable enumeration* $\{T_s\}_{s \geq 0}$ *such that* $|T_0| = 1$ *and such that for all* $s \geq 0$ *if* $\tau \in T_{s+1} - T_s$ *then* τ *extends a leaf of* T_s *(and such that a finite number of strings are enumerated at any stage).*

Definition 1.7. *We say that* $T \subseteq 2^{<\omega}$ *has* ***bounded branching*** *if there exists* n *such that every* $\tau \in T$ *has at most* n *successors in* T.

Definition 1.8. *We say that* T *is* ***Ψ-splitting*** *if whenever* $\tau, \tau' \in T$ *are incompatible,* $\Psi(\tau)$ *and* $\Psi(\tau')$ *are incompatible. We say that* A *satisfies the (bounded branching)* ***splitting tree property*** *if whenever* $A \leq_T \Psi(A)$, A *lies on a c.e.* Ψ-*splitting tree (with bounded branching).*

Definition 1.9. *For any* $\tau \in 2^{<\omega}$ *if* $|\tau| > 0$ *we define* τ^- *to be the initial segment of* τ *of length* $|\tau| - 1$, *and otherwise we define* $\tau^- = \tau$. *Given any Turing functional* Ψ *we define* $\hat{\Psi}$ *as follows. For all* τ *and all* n, $\hat{\Psi}(\tau; n) \downarrow = x$ *iff the computation* $\Psi(\tau; n)$ *converges in* $< |\tau|$ *steps,* $\Psi(\tau; n) = x$ *and* $\hat{\Psi}(\tau^-; n') \downarrow$ *for all* $n' < n$.

Definition 1.10. *We say that* $\tau \in T$ *is of* ***level n*** *in* T *if there exist precisely* n *proper initial segments of* τ *in* T. *We say that* T *is of* ***level at least n*** *if all leaves of* T *are of level at least* n *in* T. *We say that* T *is of* ***level n*** *if all leaves of* T *are of level* n *in* T.

Unfortunately c.e. traceability does not relate in such a tidy way where the minimal degrees are concerned. Gabay [YG] has shown that there are minimal degrees with strong minimal cover and which are not c.e. traceable. Since the set that Gabay constructs in this proof satisfies the splitting tree property and since any set satisfying the bounded branching splitting tree property is c.e. traceable, it follows that there exist minimal degrees containing a set which satisfies the splitting tree property which do not contain a set satisfying the bounded branching splitting tree property. In order to see that any set A satisfying the bounded branching splitting tree property is c.e. traceable, let p dominate all functions of the form $g(n) = m^n$. If A lies on a c.e. $\hat{\Psi}$-splitting tree in which each string has at most m successors, then $\Psi(A; n)$ is contained in the set $\{\hat{\Psi}(\tau; n) : \tau \text{ is of level } n + 1\}$ and this set is of size at most m^n. It is therefore clear how we may define h so as to satisfy the definition of c.e. traceability.

It is the aim of this paper to further our understanding of the issues surrounding Yates' question, in particular where the hyperimmune-free degrees are concerned.

Definition 1.11. *$A \subseteq \omega$ is **hyperimmune-free** if for every $f \leq_T A$ there exists a computable function h which dominates f (or equivalently which majorizes f) i.e. such that $h(n) \geq f(n)$ for all but finitely many n. In his new book Soare will introduce the terminology **0-dominated** in place of hyperimmune-free. Preferring this terminology, we shall adopt it in what follows.*

Definition 1.12. *We say non-empty $T \subseteq 2^{<\omega}$ is **perfect** if each $\tau \in T$ has at least two successors in T.*

In some ways the **0**-dominated degrees and the minimal degrees may be regarded as quite intimately related—the standard constructions, at least, are very similar and provide many of the same restrictions. The most basic form of minimal degree construction produces a set which satisfies the perfect splitting tree property and it may be observed that every such set is, in fact, of **0**-dominated degree. In order to see this we can argue as follows. Given f computable in A and which is incomputable, let $\Psi(A) = f$. If A lies on a perfect c.e. $\hat{\Psi}$-splitting tree then for every n, $f(n)$ is included in the values $\hat{\Psi}(\tau; n)$ for those τ of level $n + 1$ in this tree.

In what follows all notations and terminologies will either be standard or explicitly defined. For an introduction to the techniques of minimal degree construction we refer the reader to any one of [RS], [BC2], [ML], [PO1] (this paper requires knowledge only of Spector forcing).

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2. Bushy trees

Since the splitting tree technique is very much the standard approach to minimal degree construction it is natural to ask the generality of this method. Given that it seems, in the very least, to be difficult to construct a

minimal degree with no strong minimal cover using the splitting tree technique it is an obvious question to ask whether we should be looking for alternative methods of minimal degree construction. The simple observation of this section is that the sets of minimal degree can, in fact, be completely characterized in terms of **weakly splitting trees**, or in terms of c.e. trees with **delayed splitting**. These kinds of splitting trees, then, provide a perfectly general method of minimal degree construction. While it remains to be shown that their use will really enable one to do anything that the use of standard splitting trees will not, we shall argue later that the use of these kinds of trees may be necessary in constructing a negative solution to Yates' question—or, at least, in constructing such a degree which is also **0**-dominated. The result corresponding to theorem 2.1 for the sets of degree below $\mathbf{0}'$ appeared in Odifreddi [PO2], who directly transposed techniques previously described by Chong [CC] while working in α-recursion theory.

Definition 2.1. *We say that $T \subseteq 2^{<\omega}$ is a **weakly Ψ-splitting tree** if it is a c.e. subset of $2^{<\omega}$ and there exist partial computable $\phi, \psi : 2^{<\omega} \to \omega$ such that:*

(1) *if $\tau \in T$ then $\phi(\tau) \downarrow < |\tau|$ and $\psi(\tau) \downarrow < |\Psi(\tau)|$,*
(2) *whenever $\tau, \tau' \in T$ are incompatible below $\min\{\phi(\tau), \phi(\tau')\}$ they Ψ-split below $\min\{\psi(\tau), \psi(\tau')\}$,*
(3) *if $A \in [T]$ then the set $\{\phi(\tau) : \tau \subset A, \tau \in T\}$ has no finite upper bound.*

Lemma 2.1. *For any $A \subseteq \omega$ and any Turing functional Ψ, if $\Psi(A)$ is total and $A \leq_T \Psi(A)$ then A lies on a weakly Ψ-splitting tree.*

Proof. Suppose $\Phi(\Psi(A)) = A$. In order to enumerate T run through all computations of the form $\hat{\Phi}(\hat{\Psi}(\tau))$ for all $\tau \in 2^{<\omega}$, one string at a time and in order of length. Whenever we find τ, n such that $\hat{\Phi}(\hat{\Psi}(\tau))$ is defined and agrees with τ on all arguments $\leq n$ and such that this is not the case for any $\tau' \subset \tau$ enumerate τ into T and define $\phi(\tau) = n$ and $\psi(\tau) = m$, where m is the maximum x such that $\hat{\Psi}(\tau; x)$ is used in computations $\hat{\Phi}(\hat{\Psi}(\tau); n')$ for $n' \leq n$. $\qquad\square$

Lemma 2.2. *If A lies on T which is a weakly Ψ-splitting tree and $\Psi(A)$ is total then $A \leq_T \Psi(A)$.*

Proof. Suppose we are given an oracle for $\Psi(A)$. In order to compute the initial segment of A of length n proceed as follows. Enumerate T until we find $\tau \in T$ such that $\Psi(\tau)$ agrees with $\Psi(A)$ on all arguments $\leq \psi(\tau)$, and such that $\phi(\tau) \geq n - 1$. The initial segment of τ of length n is an initial segment of A. $\qquad\square$

We therefore have:

Theorem 2.1. *A is of minimal degree iff A is incomputable and, whenever $\Psi(A)$ is total and incomputable, A lies on a weakly Ψ-splitting tree.*

Theorem 2.2 below gives a version of theorem 2.1 in more familiar form. In order to get a perfectly general method of minimal degree construction all we need do is delay splitting by one level:

Definition 2.2. *We say that $T \subseteq 2^{<\omega}$ is a delayed Ψ-splitting c.e. tree if T is a c.e. tree and whenever $\tau_0, \tau_1 \in T$ are incompatible (denoted $\tau_0|\tau_1$) any $\tau_2, \tau_3 \in T$ properly extending τ_0 and τ_1 respectively are a Ψ-splitting.*

Theorem 2.2. *A is of minimal degree iff A is incomputable and, whenever $\Psi(A)$ is total and incomputable, A lies on a delayed Ψ-splitting c.e. tree.*

Proof. (Sketch) The direction from right to left is easy (note that, if whenever $\Psi(A)$ is total and incomputable A lies on a delayed Ψ-splitting c.e. tree, then A also lies on a delayed $\hat{\Psi}$-splitting c.e. tree) so we are left to prove the direction from left to right. So suppose given $A \subseteq \omega$ such that $\Psi(A)$ is total and $A \leq_T \Psi(A)$. Let T be the weakly splitting tree as defined in the proof of lemma 2.1, but with also the empty string as the single member of level 0. First we define T' as follows. The strings of level $n \leq 1$ in T' are the strings of level $n \leq 1$ in T. Suppose we have already defined the strings of level $n \geq 1$ in T'. For each $\tau \in T'$ of level n in T' the strings extending τ in T' of level $n + 1$ in T' are those strings extending τ in T and which are of level $|\tau|$ in T. It remains to show that we can take T'' such that for any n the strings of level n in T'' are strings of level n in T', such that $A \in [T'']$ and which is a c.e. tree. In order to see this consider, during the enumeration of T', the enumeration of axioms for Φ which attempts to map every set lying on T' to a 1-generic. Since A is if minimal degree and therefore does not bound a 1-generic there will exist a least i such that for all $\tau \subset A$, $\Psi_i(\Phi(\tau); i) \uparrow$ and there exists $\sigma \supset \Phi(\tau)$ with $\Psi_i(\sigma; i) \downarrow$. So long as the axioms for Φ are enumerated in a reasonably intelligent way we may then define T'' by enumerating strings of level $n + 1$ (for sufficiently large n) into this tree extending τ of level n only when we find $\sigma \supset \Phi(\tau)$ with $\Psi_i(\sigma; i) \downarrow$, and by insisting that no successors of τ should be enumerated at any subsequent stage. $\square$

3. Π_1^0 CLASSES AND THE CUPPING PROPERTY

Definition 3.1. *For any $T \subseteq \omega^{<\omega}$ we define $D(T)$ (the **downwards closure** of T) to be the set of all τ for which there exists $\tau' \supseteq \tau$ in T. We say that $T \subseteq \omega^{<\omega}$ is **downwards closed** if $D(T) = T$.*

Definition 3.2. *Downwards closed computable $T \subseteq \omega^{<\omega}$ is said to be **highly computable** if there exists a partial computable function f such that, for any $\tau \in T$, τ has at most $f(\tau)$ successors in T. A subset X of ω^ω is a Π_1^0 **class** if $X = [T]$ for some computable downwards closed T, and if T is highly computable then X is said to be a **computably bounded (c.b.)** Π_1^0 **class**.*

In what follows, it may be assumed that any Π_1^0 class referred to is computably bounded.

Theorem 3.1 (Jockusch and Soare)**.** *The **0**-dominated basis theorem: every non-empty Π_1^0 class contains a member of **0**-dominated degree.*

Definition 3.3. *A degree **a** is PA if it is the degree of a complete extension of Peano Arithmetic. Equivalently, **a** is PA if every non-empty Π_1^0 class contains a member of degree $\leq$ **a**.*

Given the similarity between the standard constructions of **0**-dominated and minimal degrees, perhaps the strongest result on the negative side of Yates' question is that there exists a **0**-dominated degree which satisfies the cupping property (and so does not have a strong minimal cover). This follows directly from the theorem of Kucera's that the PA degrees satisfy the cupping property using the **0**-dominated basis theorem, since there exists a non-empty Π_1^0 class every member of which is of PA degree.

Theorem 3.2 (Kucera [AK]). *The PA degrees satisfy the cupping property. Equivalently, there exists a non-empty Π_1^0 class every member of which is of degree which satisfies the cupping property.*

We give here a simple proof of theorem 3.2, the hope being that this alternative proof (which doesn't require reasoning within PA like the original) may be more flexibly extended in order to give stronger results. The point of this proof is not so much that it is the shortest possible, but rather that it very effectively exposes the combinatorial arguments which lie at the heart of issues surrounding the cupping property and the construction of strong minimal covers. This proof has already provided the intuition behind the constructions appearing in [AL1] and [AL2] and can be expected to have further applications.

Definition 3.4. *We let λ denote the string of length 0. $T \subseteq 2^{<\omega}$ with a single element of level 0 is **2-branching** if every $\tau \in T$ has two precisely two successors in T. We say that T is **2-branching below level n** if all $\tau \in T$ which are of level $n' < n$ in T have precisely two successors in T.*

Alternative proof of theorem 3.2. We observe first that if $A \subseteq 2^\omega$ satisfies the property that there exists a 2-branching $T \leq_T A$ such that if $C \in [T]$ then $A \not\leq_T C$, then $deg(A)$ satisfies the cupping property. Given B of degree strictly above A we define $C <_T B$ such that $C = \bigcup_n \sigma_n$. Let σ_0 be the string of level 0 in T and for all $n > 0$ let σ_n be the right successor of σ_{n-1} in T if $B(n-1) = 1$ and let σ_n be the left successor otherwise. Then $B \not\leq_T C$ since C lies on T and it is clear that B is computable given oracles for A and C.

In order to construct a non-empty Π_1^0 class every member of which is of degree which satisfies the cupping property, then, it suffices to construct downwards closed computable $\Pi \subseteq 2^{<\omega}$ such that $[\Pi]$ is non-empty and such that for every $A \in [\Pi]$ there exists 2-branching $T^A \leq_T A$ which satisfies the property that if $C \in [T^A]$ then $A \not\leq_T C$. In order to ensure that $A \not\leq_T C$ for any $C \in [T^A]$, we shall construct Ψ such that $\Psi(A) \not\leq_T C$. In particular we shall construct Π and Ψ so that $[\Pi]$ is non-empty and so as to satisfy every requirement:

$$\mathcal{N}_i : (A \in [\Pi] \wedge C \in [T^A]) \to (\Psi_i(C; i) \neq \Psi(A; i)).$$

For τ in Π we shall define values T^τ. If A is an infinite path through Π then T^A will be defined to be $\bigcup \{T^\tau : T^\tau \downarrow, \tau \subset A\}$.

So let us consider first how to satisfy a single requirement $\mathcal{N}_0$. We wish to construct Π such that $[\Pi]$ is non-empty and if $A \in [\Pi]$ there exists 2-branching $T^A \leq_T A$ which satisfies the property that if $C \in [T^A]$ then

$\Psi_0(C; 0) \neq \Psi(A; 0)$. The most primitive intuition here is as follows; if we are given four strings and we colour these strings with two colours then there exists some colour such that at least two strings are not that colour (okay so actually we only need three strings, but it convenient here to do everything in powers of two).

Now we extend this idea. Let T be the set of finite binary strings which are of even length, the important point here being that T is 4-branching. We let $T(n)$ denote the strings in T of level n in T.

Definition 3.5. *For any finite $T' \subseteq 2^{<\omega}$ and any m, an m-colouring of T' is an assignment of some $\mathrm{col}(\sigma) < m$ to each leaf σ of T'.*

Definition 3.6. *Given any $f : \omega \to \omega$ we say that non-empty T' is (T, f) **compatible** if for every n the strings of level n in T' are strings of level n in T and any string of level n in T' which is not a leaf of T' has $f(n)$ successors in T'.*

Let κ be the constant function such that for all n, $\kappa(n) = 2$. We say that T' is $(T, 2)$ compatible if it is (T, κ) compatible. The following lemma is just what we need in order to be able to satisfy the single requirement $\mathcal{N}_0$.

Lemma 3.1. *For every n and every 2-colouring of $T(n)$ there exists $d < 2$ and $(T, 2)$ compatible T' of level n such that no leaf σ of T' has $\mathrm{col}(\sigma) = d$.*

Proof. The case $n = 0$ is trivial and, in fact, we have already seen the case $n = 1$ since there are four strings in $T(1)$ and any two of these strings define a $(T, 2)$ compatible T' of level one.

So suppose the result holds for all $n' \leq n$. Given any 2-colouring of $T(n + 1)$ consider each $\sigma \in T(n)$. Each such σ has four successors in T which are strings in $T(n + 1)$. If there exists $d < 2$ such that more than two of the successors σ' of σ in T have $\mathrm{col}(\sigma') = d$ then define $\mathrm{col}(\sigma) = d$ and otherwise define $\mathrm{col}(\sigma) = 0$. This gives a 2-colouring of $T(n)$ and by the induction hypothesis there exists T' which is $(T, 2)$ compatible of level n and $d < 2$ such that no leaf σ of T' has $\mathrm{col}(\sigma) = d$. In order to define T'' which is $(T, 2)$ compatible and of level $n + 1$ and such that no leaf σ' of T'' has $\mathrm{col}(\sigma') = d$ just choose two extensions of each leaf σ of T' which are not coloured d. We have defined the 2-colouring of $T(n)$ precisely so that this is possible. $\square$

Now we see how to use this lemma in order to satisfy the first requirement. Before defining Π we define a set of strings $\Pi^\star$—these are strings which may or may not be in Π. We do not require that $\Pi^\star$ is downwards closed. Once we have defined this set we shall form Π by taking certain strings from $\Pi^\star$ and then adding strings in so that Π will be downwards closed. For every n we have to define the set $\Pi^\star(n)$ which is the set of strings in $\Pi^\star$ of level n in $\Pi^\star$, for each of these strings τ we have to define a value T^τ and we also have to ensure that $\Psi(\tau; 0) \downarrow$. The latter condition we satisfy by defining $\Psi(\tau; 0)$ for all $\tau \in \Pi^\star(1)$. We shall explain exactly how one may define $\Pi^\star$ and the other values just discussed in a moment, but the important point here is just this; we can do so in such a way that for any $d < 2$ and any $(T, 2)$ compatible T' of level $n \geq 1$ there exists $\tau \in \Pi^\star(n)$ such that $T^\tau = T'$

and $\Psi(\tau; 0) = d$. Really this is completely obvious—all you need do is to put enough strings in $\Pi^\star(n)$ so that all possibilities can be realized.

What this means is that if we define Π by taking the strings in $\Pi^\star$ except for those strings τ for which we observe that there exists $\sigma \in T^\tau$ with $\Psi_0(\sigma; 0) = \Psi(\tau; 0)$ then for every n there must exist $\tau \in \Pi^\star(n)$ which is in Π (and thus $[\Pi]$ will be non-empty). This follows because we can consider the values $\Psi_0(\sigma; 0)$ for σ in $T(n)$ to define a 2-colouring of $T(n)$. For any 2-colouring of $T(n)$ there exists $d < 2$ and a $(T, 2)$ compatible T' of level n such that no leaf of T' is coloured d. Then $\tau \in \Pi^\star(n)$ with $T^\tau = T'$ and $\Psi(\tau; 0) = d$ will be a string in Π.

Before going on to consider how we may satisfy all requirements, then, let's see how to define $\Pi^\star$ (when we are only looking to satisfy the first requirement). We define $\Pi^\star(n)$ by recursion on n. We define $\Pi^\star(0) = \{\lambda\}$ and $T^\lambda = \{\lambda\}$. Let $\{T_0, ..., T_{m-1}\}$ be the set of all $(T, 2)$ compatible T' which are of level 1. Let $\tau_0, ..., \tau_{2m-1}$ be pairwise incompatible, define $\Pi^\star(1) = \{\tau_0, ..., \tau_{2m-1}\}$ and for each $i < m$ define $T^{\tau_{2i}} = T_i$, $T^{\tau_{2i+1}} = T_i$, $\Psi(\tau_{2i}; 0) = 0$ and $\Psi(\tau_{2i+1}; 0) = 1$.

Given $\Pi^\star(n)$ for $n > 0$ we define $\Pi^\star(n+1)$ as follows. For each $\tau \in \Pi^\star(n)$ let $\{T_0, ..., T_{m'-1}\}$ be the set of all $(T, 2)$ compatible T' such that $T^\tau \subset T'$ and which are of level $n + 1$. Let $\tau_0, ..., \tau_{m'-1}$ be pairwise incompatible extensions of τ, enumerate these strings into $\Pi^\star(n+1)$ and for each $i < m'$ define $T^{\tau_i} = T_i$.

In order to satisfy every requirement $\mathcal{N}_i$ while maintaining non-empty $[\Pi]$ we must become a little more sophisticated, but the basic idea remains the same. We need a 'bushier' T and we need also to use more colours for lower priority requirements:

Definition 3.7. *For every n the set $T(n)$—the strings in T of level n— are those elements of $2^{<\omega}$ of length $\Sigma_{i<n}(i + 2)$.*

Definition 3.8. *For every i we let κ_i be defined as follows. We have $\kappa_0(0) = 4$. For all $n < i$, $\kappa_i(n) = 2$ and for all $n \geq i$, $\kappa_i(n) = 2\kappa_i(n-1)$ (if $n-1 \geq 0$). For all i we define $ncol(i) = 2^{i+1}$.*

For every i, then, the value $ncol(i)$ should be thought of as the number of colours that we use in order to satisfy the requirement $\mathcal{N}_i$. We need a new version of lemma 3.1:

Lemma 3.2. *If T_0 is (T, κ_i) compatible and of level n then for any $ncol(i)$-colouring of T_0 there exists $T_1 \subseteq T_0$ which is (T, κ_{i+1}) compatible of level n and $d < ncol(i)$ such that no leaf σ of T_1 has $col(\sigma) = d$.*

Proof. Given any fixed i we prove the result by induction on n. The base case, for those $n \leq i$, follows trivially since there are at most 2^i strings in T_0 of level n and $ncol(i) = 2^{i+1}$. So suppose that $n \geq i$ and that the result holds for n. Given any $ncol(i)$-colouring of T_0 which is (T, κ_i) compatible and of level $n + 1$ consider each string σ of level n in T_0. Such σ has 2^{n-i+2} successors σ' in T_0. If there exists some $d < ncol(i)$ such that more than half of those successors σ' have $col(\sigma') = d$ then define $col(\sigma) = d$ and otherwise define $col(\sigma) = 0$. Let T_0' be the set of strings in T_0 of level $\leq n$. We have

defined an $\mathrm{ncol}(i)$-colouring of T_0'. By the induction hypothesis there exists $T_1' \subseteq T_0'$ which is (T, κ_{i+1}) compatible of level n and $d < \mathrm{ncol}(i)$ such that no leaf σ of T_1' has $\mathrm{col}(\sigma) = d$. In order to define $T_1 \subseteq T_0$ which is (T, κ_{i+1}) compatible of level $n + 1$ such that no leaf σ of T_1 has $\mathrm{col}(\sigma) = d$, simply choose 2^{n-i+1} extensions σ' of each leaf σ of T_1' such that $\mathrm{col}(\sigma') \neq d$. We defined the colouring of T_0' precisely so that this is possible. $\qquad\square$

The intuition now runs as follows. For each i we shall define any value $\Psi(\tau; i) = d$ so that $d < \mathrm{ncol}(i)$. We are yet to define Π and $\Pi^\star$, but we will shortly do so in a manner analogous to what went before. Suppose that for some n there are no members of $\Pi^\star(n)$ in Π. Let $T_0 = \bigcup_{i \leq n} T(i)$. Then we consider those values $\hat{\Psi}_0(\sigma; 0)$ for the leaves σ of T_0 to define a 2-colouring of T_0. Lemma 3.2 then suffices to show, not only that there exists $\tau \in \Pi^\star(n)$ such that no leaf σ of T^τ has $\mathrm{col}(\sigma) = \Psi(\tau; 0)$, but that there exists a range of such values—all those τ, in fact, such that a) T^τ is a subset of some fixed $T_1 \subset T_0$ which is (T, κ_1) compatible of level n and b) such that $\Psi(\tau; 0) = d_0$ for some fixed $d_0 < 2$. Next we consider those values $\hat{\Psi}_1(\sigma; 1)$ for the leaves σ of T_1 to define a 4-colouring of T_1. Once again we apply lemma 3.2 in order to obtain $T_2 \subset T_1$ which is (T, κ_2) compatible of level n and $d_1 < 4$ such that if $\tau \in \Pi^\star(n)$, $T^\tau \subseteq T_2$ and $\Psi(\tau; 0) = d_0$, $\Psi(\tau; 1) = d_1$ then no leaf σ of T^τ has $\hat{\Psi}_0(\sigma; 0) = \Psi(\tau; 0)$ or $\hat{\Psi}_1(\sigma; 1) = \Psi(\tau; 1)$. Proceeding inductively in this way we are able to reach the required contradiction.

Once again we define $\Pi^\star(n)$ by recursion on n. We define $\Pi^\star(0) = \{\lambda\}$ and $T^\lambda = \{\lambda\}$. Given $\Pi^\star(n)$ we define $\Pi^\star(n + 1)$ as follows. For each $\tau \in \Pi^\star(n)$ let $\{T_0, ..., T_{m-1}\}$ be the set of all $(T, 2)$ compatible T' such that $T^\tau \subset T'$ and which are of level $n + 1$. Let $\mathrm{ncol}(n) = m'$ and let $\tau_0, ..., \tau_{mm'-1}$ be pairwise incompatible extensions of τ (these strings may be thought of as being divided into m collections of size m'), enumerate these strings into $\Pi^\star(n + 1)$ and for each $i < m$ proceed as follows: for each j with $m'i \leq j < m'(i + 1)$ define $T^{\tau_j} = T_i$ and $\Psi(\tau_j; n) = j - m'i$.

Lemma 3.3. *For any n, any $f \in \omega^{<\omega}$ of length n and any $(T, 2)$ compatible T' of level n, if it is the case that for all $i < n$, $f(i) < \mathrm{ncol}(i)$ then there exists $\tau \in \Pi^\star(n)$ such that $T^\tau = T'$ and $\Psi(\tau) = f$.*

Proof. The proof is not difficult and is left to the reader. $\qquad\square$

We are now ready to define Π. We do so in stages.

Stage 0. We define $\Pi_0 = \{\lambda\}$.

Stage s + 1. Initially Π_{s+1} is empty. For every string τ which is a leaf of Π_s we proceed as follows. We shall have that τ is a string of level s in $\Pi^\star$. If it is not the case that there exists $\sigma \in T^\tau$ and $i < s$ such that $\hat{\Psi}_i(\sigma; i)$ is equal to $\Psi(\tau; i)$ then enumerate every successor of τ in $\Pi^\star$ into Π_{s+1}, together with all initial segments of such strings.

We define $\Pi = \bigcup_s \Pi_s$.

Lemma 3.4. *The class $[\Pi]$ is non-empty.*

Proof. Suppose towards a contradiction that n is the least such that there exist no strings of level n in $\Pi^\star$ which are in Π (it follows from König's lemma

that such a contradiction suffices to give the result). Let $T_0 = \bigcup_{i \leq n} T(i)$. We proceed inductively to define T_i and d_{i-1} for each $i \leq n, i \geq 1$. Given T_i which is (T, κ_i) compatible of level n we let the values $\hat{\Psi}_i(\sigma; i)$ for those σ which are leaves of T_i define an ncol(i)-colouring of T_i (we may assume that for any σ and any i, if $\Psi_i(\sigma; i)$ is defined then it is less than ncol(i)—otherwise we may regard this computation as non-convergent). We then apply lemma 3.2 in order to find $T_{i+1} \subseteq T_i$ which is (T, κ_{i+1}) compatible of level n and $d_i <$ncol(i) such that no leaf σ of T_{i+1} has col$(\sigma) = d_i$. Let the string f of length n be defined such that for all $i < n$, $f(i) = d_i$. By lemma 3.3 there exists $\tau \in \Pi^\star(n)$ such that $T^\tau = T_n$ and $\Psi(\tau) = f$. Then τ is an element of $\Pi^\star(n)$ which is in Π.

$\square$

Lemma 3.5. *If $A \in [\Pi]$ then deg(A) satisfies the cupping property.*

Proof. Suppose that $A \in [\Pi]$ and that there exists $C \in [T^A]$ such that $\Psi_i(C) = \Psi(A)$. Then, in particular, there exists σ in T^A which is an initial segment of C and such that $\hat{\Psi}_i(\sigma; i) = \Psi(A; i)$. Let σ be the shortest, suppose that σ is of level n in T^A and let s be the least stage such that $s > i + 1$ and $s > n$. At stage s in the construction of Π we shall ensure that $A \notin [\Pi]$. This gives us the required contradiction. $\square$

What can we say about Π_1^0 classes every member of which is of degree with strong minimal cover? Of course, it is a trivial matter to define a Π_1^0 class which contains a single (computable) member and so which satisfies this condition. On the other hand, there cannot exist such a class of positive measure since every such class contains a member of every random degree, and therefore an element of every degree above $\mathbf{0}'$. In a sense, then, the following theorem is the strongest we could hope for:

Theorem 3.3. *There exists a non-empty Π_1^0 class with no computable members, every member of which is of degree with strong minimal cover.*

Proof. We shall construct downwards closed Π such that $[\Pi]$ is non-empty and contains no computable elements, and such that every member of this class is c.e. traceable. Let p be a computable function which dominates every function p_i such that $p_i(n) = 2^{n+i}(n + i)!$—why it that we consider this particular function will become clear subsequently. We shall act in order to satisfy the requirements:

$\mathcal{C}_i$: $(A \in [\Pi]) \wedge (\Psi_i(A) \text{ is total}) \Rightarrow$ there exists computable h such that $|W_{h(n)}| \leq p(n)$ and $\Psi_i(A; n) \in W_{h(n)}$ for all n.

$\mathcal{P}_i$: If $A \in [\Pi]$ then $A \neq \Psi_i(\emptyset)$.

So let us consider first how to satisfy the requirement $\mathcal{C}_0$. At the end of each stage $s + 1$ we will add each string τ of length $s + 1$ into Π unless τ has already been declared terminal. Various strings in Π will be declared as 'nodes' and will be allocated modules of two kinds. A $\mathcal{C}$ module of the form $(0, n)$ is concerned with satisfaction of the requirement $\mathcal{C}_0$. If allocated to the string τ it searches at each stage $s + 1$ for $\tau' \supseteq \tau$ with two incompatible extensions of length s which have not been declared terminal, and such that the computation $\Psi_0(\tau'; n)$ converges in at most s steps. When the

module $(0, n)$ finds such τ' we say that the module 'acts'. All strings which are proper extensions of τ are declared not to be nodes and all modules are removed from these strings. It chooses incompatible τ_0, τ_1 extending τ' which are of length s and which have not been declared terminal. It declares all extensions of τ which are not compatible with either τ_i to be terminal and allocates the module $(0, n+1)$ to each of the τ_i which are now declared to be nodes. Thus if $A \in [\Pi]$ extends τ the module $(0, n)$ restricts $\Psi_0(A; n)$ to at most one possible value. The module acts only once (for given τ). The modules $(0, n+1)$ allocated to each τ_i then combine in order to restrict $\Psi_0(A; n+1)$ to at most two possible values, and so on.

In order to satisfy the requirement $\mathcal{C}_1$ we shall proceed in just the same way except that, whereas we begin to allocate modules for the sake of requirement $\mathcal{C}_0$ at the node λ, now we only begin to allocate modules for the sake of this requirement at nodes of level 1 (nodes which have precisely one proper initial segment as a node). In general we begin to allocate modules for the sake of satisfying the requirement $\mathcal{C}_i$ at nodes of level i.

The $\mathcal{P}$ module (i) allocated to the node τ is concerned with satisfaction of the requirement $\mathcal{P}_i$. We shall have that τ is a node of level i. If $\tau' \supset \tau$ is a node and there does not exist any node τ'' with $\tau \subset \tau'' \subset \tau'$, then τ' is called a 'successor node' of τ. If (i) has not already acted then τ will have two successor nodes, τ_0 and τ_1 say. If it finds at any stage $s+1$ that $\Psi_i(\emptyset)$ extends one of these successor nodes τ_i then the module 'acts' (just the once) by declaring all extensions of τ which are not compatible with $\tau_{|1-i|}$ as terminal and not to be nodes, and by removing all modules from these strings.

The only point of any difficulty in this construction is that the following kind of injury may occur. Let's suppose that $\tau' \supset \tau$ are both nodes. A module allocated to τ' may act so as to restrict the number of possible values $\Psi_i(A; n)$ for all $A \in [\Pi]$ extending τ' and it may then be the case that a module allocated to τ acts and defines a node or nodes of the form τ'' such that $\Psi_i(\tau''; n) \uparrow$. The simple remedy to this apparent problem is to observe that we can easy bound the number of injuries that can occur.

We are ready to define the construction.

The module (i, n) allocated to τ. At stage $s+1$ the module searches for $\tau' \supseteq \tau$ with two incompatible extensions of length s which have not been declared terminal, and such that the computation $\Psi_i(\tau'; n)$ converges in at most s steps. If there exists such τ' we say that the module 'acts'. In this case it carries out the following instructions:

- All strings which are proper extensions of τ are declared not to be nodes and all modules are removed from these strings.
- It chooses incompatible τ_0, τ_1 extending τ' which are of length s and which have not been declared terminal.
- The node τ will have been allocated a module $(j-1)$, here $j-1$ will be the level of τ as a node. The module (i, n) then declares all extensions of τ which are not compatible with either τ_i to be terminal and allocates the modules (j) and $(j', j-j')$ for each $j' \leq j$ to each of the τ_i which are now declared to be nodes.

- The module enumerates the tuple $(i, n, \Psi_i(\tau'; n))$.

The instructions for the module (i) allocated to τ are precisely as previously described.

Stage 0. Enumerate λ into Π, declare λ to be a node and allocate it the modules $(0, 0)$ and (0).

Stage s + 1. Run all modules allocated to all nodes prior to this stage, in order of the level of the node. All modules allocated to each node can be run in any fixed order, but we only allow one module allocated to each node to act at any given stage. For each string of τ of length $s + 1$ which has not been declared terminal proceed as follows 1) enumerate τ into Π, 2) declare τ to be a node, 3) if τ is a node of level n then allocate the modules (n) and $(n', n - n')$ for every $n' \leq n$ to τ.

The verification. Let's say that τ is a 'final node' if there is a point of the construction at which it is declared to be a node and after which it never subsequently declared not to be a node. If $\tau' \supset \tau$ are both final nodes and there does not exist any final node τ'' with $\tau \subset \tau'' \subset \tau'$, then τ' is called a 'final successor node' of τ. We say that τ is a final node of level n if it is a final node and has precisely n proper initial segments which are final nodes. We first show that if τ is a final node of level n then:

(1) τ has at at least one final successor node.
(2) if τ' is a final successor node of τ then $\tau' \not\subset \Psi_n(\emptyset)$.
(3) if $A \in [\Pi]$ extends τ then it extends a final successor node of τ.

This suffices to show that $[\Pi]$ is non-empty and that any $A \in [\Pi]$ is incomputable. So suppose that τ is a final node of level n. Then subsequent to the last stage in the construction at which τ is declared to be a node, stage s say, no $\mathcal{C}$ module allocated to a node which is a proper initial segment of τ acts. Let $s' > s$ be the last stage at which any $\mathcal{C}$ module allocated to τ acts, and if there exists no such then let $s' = s + 1$. By the end of stage s', τ has precisely two successor nodes, τ_0 and τ_1 say and these are the only strings in Π extending τ of length $|\tau_0|$ (we can assume that a $\mathcal{P}$ module allocated to a node does not act until two stages after the node has been declared). These two nodes are both final and satisfy the property that they are not initial segments of $\Psi_n(\emptyset)$ unless the module (n) allocated to τ subsequently acts so as to declare some τ_i terminal. In this case the remaining successor node satisfies the required property.

We are left to show that if $A \in [\Pi]$ then A is c.e. traceable. Consider the requirement $\mathcal{C}_i$. Observe first that for all n, A extends a final node which is allocated the module (i, n) so that if $\Psi_i(A; n)$ is not equal to some value d for which we enumerate a tuple (i, n, d) then $\Psi_i(A; n)$ is undefined (it is a basic result of Π^0_1 classes that since $[\Pi]$ contains no computable members it must be perfect, so that if $\Psi_i(\tau; n) \downarrow$ for some $\tau \subset A$ then we will eventually be able to find the incompatible extensions required for the module to act). Now if a tuple of this form is enumerated then it is enumerated by a node of level $n + i$ and each such node can only enumerate at most one tuple of this form—each time a string is declared to be a node we consider this to be a new node. It therefore suffices to show that for every n there exist at

most $2^n(n+1)!$ many nodes declared of level n. If we do this then we will have shown that for every n there at most $2^{n+i}(n+i+1)!$ values d for which we enumerate a tuple (i, n, d). If we define $W_{h'(n)}$ to be this set of values then for almost all n we have $|W_{h'(n)}| \leq p(n)$ so that a finite adjustment to h' yields the function h required for satisfaction of the requirement $\mathcal{C}_i$. We proceed by induction. The case $n = 0$ is clear, so suppose that the result holds for all $n' \leq n$ so that there are at most $2^n(n+1)!$ nodes defined of level n. Each such node is allocated $n + 1$ different $\mathcal{C}$ modules and can therefore have at most $2(n+2)$ different successors during the construction. A simple calculation $2(n+2).2^n(n+1)! = 2^{n+1}(n+2)!$ completes the induction step. $\qquad\square$

4. The FPF degrees

Definition 4.1. *We say that $A \subseteq \omega$ is of fixed point free (FPF) degree if there exists $f \leq_T A$ such that $\Psi_n(\emptyset) \neq \Psi_{f(n)}(\emptyset)$ for all n. We say f is DNR if for all n if $\Psi_n(\emptyset; n) \downarrow$ then $f(n) \neq \Psi_n(\emptyset; n)$.*

Definition 4.2. *We say that $T \subseteq 2^{<\omega}$ is a **weak c.e. tree** if it has a computable enumeration $\{T_s\}_{s \geq 0}$ such that $T_0 = \{\lambda\}$ and such that for all $s \geq 0$, $|T_{s+1}| - |T_s| \leq 1$ and if τ is in $T_{s+1} - T_s$ then τ is a leaf of T_{s+1}.*

Definition 4.3. *We say a set of strings is **prefix-free** if its elements are pairwise incompatible. Given $T \subseteq 2^{<\omega}$ and $\tau \in T$ we define $T_\tau = \{\tau' \in T : \tau' \supseteq \tau\}$ and we define $|\tau|_T$ to be the level of τ in T. We say that $T' \subseteq T$ is T-**thin** if $\lambda \in T'$ and for every $\tau \in T'$ and every prefix-free $\Lambda \subseteq T'_\tau$ we have that $\Sigma_{\tau' \in \Lambda} 2^{-|\tau'|_{T_\tau}} \leq 1$.*

Definition 4.4. *We let $C(\sigma)$ denote the plain Kolmogorov complexity of σ.*

Theorem 4.1 ([KMS])**.** *If A is of FPF degree then there exists $g \leq_T A$ such that for all n, $C(A \restriction g(n)) > n$.*

Theorem 4.2. *The following conditions on $A \subseteq \omega$ are equivalent:*
(1) *For any weak c.e. tree T, if $A \in [T]$ then there exists c.e. $T' \subseteq 2^{<\omega}$ which is T-thin and such that $A \in [T']$ (T' is not required to be a weak c.e. tree).*
(2) *There exists a computable p such that for every $f \leq_T A$ there exists a computable h such that $|W_{h(n)}| \leq p(n)$ for all $n \in \omega$ and for infinitely many n we have $f(n) \in W_{h(n)}$.*
(3) *For every $f \leq_T A$ there exists a computable h such that $|W_{h(n)}| \leq n$ for all $n \in \omega$ and for infinitely many n we have $f(n) \in W_{h(n)}$.*
(4) *The degree of A is not FPF.*

Proof. $(1) \Rightarrow (2)$ Given A which satisfies (1) and $f = \Psi(A)$ let T be defined as follows. The string of level 0 in T is the empty string and for every $n > 0$ the strings of level n in T are those strings τ such that $\hat{\Psi}(\tau)$ is of length n and such that this is not the case for any $\tau' \subset \tau$. If T' is c.e. and T-thin with $A \in [T']$ then for every n there exist at most 2^n strings in T' of level n in T. Define $W_{h(n)}$ to be the set of values $\Psi(\tau)(n)$ for those τ in T' of level $n + 1$ in T and define $p(n) = 2^{n+1}$. Note that p does not depend upon Ψ.

(2)$\Rightarrow$(3) We proceed just as in [TZ] (where the same argument was made concerning a strengthening of the condition of c.e. traceability). So suppose that A satisfies (2) and that we are given $f \leq_T A$. We can assume that $p(0) = 0$ and that for all n, $p(n + 1) > p(n)$. For every n let $k(n)$ be the greatest m such that $p(m) \leq n$. For every n let $k'(n)$ be the least m such that $k(m) > n$. Define $f'(n)$ to be an effective coding of $f \restriction k'(n)$ and let h be such as to satisfy condition (2) with respect to f' and p. Defining $W_{h'(n)}$ to be the set of values $\tau(n)$ for those τ whose codes are in $W_{h(k(n))}$ suffices to show that (3) is satisfied with respect to f.

(3)$\Rightarrow$(1) Suppose that A which satisfies (3) lies on some weak c.e. tree T. First we define $T^\star$ as follows; for every n the strings of level n in $T^\star$ are the strings of level $\Sigma_{i \leq n} 2i$ in T. For all n define $f(n)$ to be (some effective coding of) the initial segment of A which is of level n in $T^\star$. Let h be such as to satisfy (3) with respect to f. We can assume that if $m \in W_{h(n)}$ then m codes a string of level n in $T^\star$. Then T' which is the empty string together with all strings whose codes are in $\bigcup_n W_{h(n)}$ is T-thin with $A \in [T']$. In order to see this suppose that $\tau \in T'$ and let Λ be the strings in T' which properly extend τ. We show that $\Sigma_{\tau' \in \Lambda} 2^{-|\tau'|_{T_\tau}} < 1$. Suppose τ is of level n in $T^\star$. For every $i > 0$ there are at most $n + i$ strings in T' which are of level $n + i$ in $T^\star$ (and extend τ) and each such string is of level at least $2(n + i)$ in T_τ. Then $\Sigma_{\tau' \in \Lambda} 2^{-|\tau'|_{T_\tau}} \leq \Sigma_{i > 0}(n + i)2^{-2(n+i)} < 1$.

(4)$\Rightarrow$(3) It is well known that A is of FPF degree iff A computes a DNR function, so if A is not of FPF degree then for any $f \leq_T A$ there exist infinitely many n with $\Psi_n(\emptyset; n) \downarrow = f(n)$. For all n we can therefore define $W_{h(n)} = \{\Psi_n(\emptyset; n)\}$ if $\Psi_n(\emptyset; n) \downarrow$ and $W_{h(n)} = \emptyset$ otherwise.

(3)$\Rightarrow$(4) We suppose we are given A which is of FPF degree and which satisfies (3) and then produce a contradiction. In order to do so we extend an argument provided in [KMS]. By theorem 4.1 we may let $g \leq_T A$ be such that for all n, $C(A \restriction g(n)) > n$. For all n define $f(n)$ to be some effective coding of $A \restriction g(n)$ and let h be witness to the fact that (3) is satisfied with respect to f. There exists c such that, for all n with $f(n) \in W_{h(n)}$ we have $C(A \restriction g(n)) \leq 3 \ln n + c$ which gives the required contradiction. In order to see this observe that in order to specify $f(n)$ (and so $A \restriction g(n)$) whenever $f(n) \in W_{h(n)}$ all we need is a string of the form $\tau_0 \tau_1$ where in order to form τ_0 we write n in binary notation and then put a 0 after each bit except the last after which we put a 1 (so that one can see where the coding of n finishes and the coding of the position of $f(n)$ within $W_{h(n)}$ starts), and where in order to form τ_1 we just write m in binary notation where $f(n)$ is the mth element enumerated into $W_{h(n)}$.

$\square$

Theorem 4.3. *Every **0**-dominated degree which is not FPF has a strong minimal cover.*

Proof. See sections 5 and 6. $\square$

Of course, the question as to whether or not there exists a minimal degree which is FPF was another longstanding question concerning minimal degrees and it is interesting that, at least where the **0**-dominated degrees are

concerned, these two questions now seem to be related. In an unpublished paper [MK] Kumabe has constructed a FPF minimal degree.

Theorem 4.4. *If A satisfies the splitting-tree property then A is not of FPF degree.*

Proof. Suppose that A satisfies the splitting-tree property and for some weak c.e. tree T we have that $A \in [T]$. We define Ψ by enumerating axioms as follows: for every string τ of level n in T we enumerate the axiom $\Psi(\tau) = \tau \upharpoonright n$. Let T' be a c.e. Ψ-splitting tree such that $A \in [T']$. We can assume that $T' \subseteq T$ and $\lambda \in T'$. Then T' is c.e. and T-thin. $\square$

Let us consider for a moment exactly what theorem 4.4 means. It is certainly the case that we may interpret this theorem in a *constructive* sense. Theorem 4.4 tells us, for example, that there exist **0**-dominated degrees which are not c.e. traceable and not FPF. This can be seen through an analysis of Gabay's proof that there exists a minimal degree with strong minimal cover and which is not c.e. traceable. The techniques developed suffice to give a set of minimal degree which is not c.e. traceable and which satisfies the perfect splitting tree property. Comments made in the introduction to this paper then suffice to show that this degree is **0**-dominated and theorem 4.4 suffices to show that it is not FPF. One might also try to interpret theorem 4.4, though, as saying that the standard splitting tree technique cannot be used in order to construct a minimal degree which is FPF, and that in order to do so the use of delayed splitting trees is necessary. Of course, the functional Ψ defined in the proof of the theorem is so trivial that this case has not yet been made. If it is the case that *whenever* $A \leq_T \Psi(A)$, A lies on a c.e. Ψ-splitting tree then A is not of FPF degree, but one might suppose that it is possible to proceed using standard splitting trees while ignoring certain Ψ when for some reason it will obviously not be problematic to do so. It is interesting to observe, anyway, that in constructing a minimal degree which is FPF, Kumabe uses delayed splitting c.e. trees. In light of theorem 4.3 it seems reasonable to suggest that delayed splitting c.e. trees are likely to be necessary in the construction of a minimal degree with no strong minimal cover—or, at least, in constructing such a degree which is also **0**-dominated.

Since no 1-generic is FPF it follows from theorem 4.3 that any **0**-dominated degree which is bounded by a 1-generic has strong minimal cover. It therefore seems of relevance to know whether there exist non-trivial examples of such degrees. The following definition is due to Chong and Downey.

Definition 4.5. $T \subseteq 2^{<\omega}$ *is said to be* Σ_1 ***dense*** *in A if:*

- *no element of T is an initial segment of A,*
- *for any c.e. $T' \subseteq 2^{<\omega}$ such that $A \in [D(T')]$, some member of T' extends a member of T.*

Chong and Downey [CD1], [CD2] have shown that any set A is computable in a 1-generic iff there is no c.e. set of strings T which is Σ_1 dense in A.

Using this characterization they were able to show that there is a minimal degree below **0**$'$ which is bounded by a 1-generic below **0**$''$, and also that

there is a minimal degree below $\mathbf{0}'$ which is not bounded by any 1-generic. The following theorem has also been proved independently in a joint paper by Downey and Yu [DY].

Theorem 4.5. *There are $\mathbf{0}$-dominated degrees which are not bounded by any 1-generic and $\mathbf{0}$-dominated degrees ($\neq \mathbf{0}$) which are bounded by a 1-generic degree.*

Proof. That there exist $\mathbf{0}$-dominated degrees which are not bounded by any 1-generic follows from the fact that there exist $\mathbf{0}$-dominated degrees which are FPF. In order to show that there exists a $\mathbf{0}$-dominated degree ($\neq \mathbf{0}$) which is bounded by a 1-generic we may proceed almost exactly as in [CD1] in order to construct A of $\mathbf{0}$-dominated minimal degree such that there is no c.e. set of strings T which is Σ_1 dense in A. We construct a set A of minimal degree below $\mathbf{0}''$ which lies on perfect splitting trees. At each stage $s+1$, having defined $\tau \supseteq A_s$ of which A_{s+1} will be an extension and a tree T_{s+1} which satisfies the property that if $A \in [T_{s+1}]$ then the sth minimality requirement will be satisfied, we then ask whether there exists $\tau' \in \Pi_s$—the sth c.e. set of strings—which is extended by a string in T_{s+1} extending τ. If so then we may define A_{s+1} so as to extend such τ' and otherwise the strings in T_{s+1} extending τ are a c.e. set of strings which is witness to the fact that Π_s is not Σ_1 dense in A. $\square$

We close this section by observing that another technique of minimal degree construction always produces minimal degrees with a strong minimal cover.

Definition 4.6. *For any $\Pi \subseteq 2^{<\omega}$ we define $B([\Pi])$, the **Cantor-Bendixson derivative of** $[\Pi]$, to be the set of non-isolated points of $[\Pi]$ according to the Cantor topology. The **iterated Cantor-Bendixson derivative** $B^\alpha([\Pi])$ is defined for all ordinals α by the following transfinite induction. $B^0([\Pi]) = [\Pi]$, $B^{\alpha+1}([\Pi]) = B(B^\alpha([\Pi]))$ and $B^\lambda([\Pi]) = \bigcap_{\alpha<\lambda} B^\alpha([\Pi])$ for any limit ordinal λ.*

Definition 4.7. *A set A has **Cantor-Bendixson rank** α if α is the least ordinal such that for some Π_1^0 class $[\Pi]$, $A \in B^\alpha([\Pi]) - B^{\alpha+1}([\Pi])$.*

Theorem 4.6 (Cenzer, Smith [CS]). *If $B \leq_{tt} A$ and A has rank n then B has rank $m \leq n$.*

Theorem 4.7 (Owings [JO]). *If $rk(B) = rk(A \oplus B)$ then $A \leq_T B$.*

Theorem 4.8 (Downey [RD]). *There exists a set of $\mathbf{0}$-dominated degree and which is of rank one.*

Theorem 4.9. *If A is of rank one and is of $\mathbf{0}$-dominated degree then it is of minimal degree.*

Proof. Suppose that A is of $\mathbf{0}$-dominated degree, that A is of rank 1, and that there exists incomputable $B <_T A$. Generally speaking whenever $C \leq_T D$ and D is of $\mathbf{0}$-dominated degree we actually have $C \leq_{tt} D$. By theorem 4.6, then, B must be of rank 1 since to be of rank 0 would mean that B is computable. But then $A \oplus B$ is also of rank 1 so that by theorem 4.7 we have that A is computable in B which gives a contradiction. $\square$

Theorem 4.10. *If A is of **0**-dominated degree and is of rank 1 then the degree of A is not FPF and therefore has strong minimal cover.*

Proof. We suppose we are given A which satisfies the hypothesis of the theorem and we show that this set satisfies (3) of theorem 4.2. So let $[\Pi]$ be a Π_1^0 class such that A is the unique non-isolated point of $[\Pi]$. Given $f = \Psi(A)$ we may take computable g which majorizes the use function for Ψ with oracle A. For every n we consider $\Pi(n)$, the strings in Π of length n. Suppose there are m strings in this set. Since only one of these strings is an initial segment of a non-isolated point of $[\Pi]$ there exists some $\Lambda \subset \Pi(n)$ of size $m-1$ and some large n', n'' with $n'' > g(n')$, such that there do not exist n' strings in $\Pi(g(n'))$ extending a string in Λ and which have an extension in $\Pi(n'')$. In fact we can effectively find such Λ, n', n'' so that for each n, having found such values, we can define $W_{h(n')}$ to be the set of values $\Psi(\tau; n')$ for those τ in $\Pi(g(n'))$ extending a string in Λ which have an extension in $\Pi(n'')$ and we can enumerate also the string, τ_n say, which is the unique element of $\Pi(n) - \Lambda$. For each n (considered in turn) we can insist that n' is larger than any number previously mentioned during the construction and then define $W_{h(k)}$, for all $k < n'$, to be the empty set unless this value is already defined. If it is the case that for almost all n, $\tau_n \subset A$ then A is computable and otherwise we have that for infinitely many n, $f(n) \in W_{h(n)}$. $\square$

5. Constructing a strong minimal cover

In this section we shall discuss a straightforward approach to be taken in attempting to construct a strong minimal cover for any given degree. In so doing lemma 5.1 will be useful.

Definition 5.1. *We shall say that τ is **$A\oplus$-compatible** if, for all n such that $\tau(2n) \downarrow$, we have $\tau(2n) = A(n)$.*

Lemma 5.1. *Suppose $\Psi = \hat{\Psi}$. If T_0 is an A-computable 2-branching Ψ-splitting tree, then $T_1 = \{\Psi(\tau) : \tau \in T_0\}$ is an A-computable 2-branching tree. Let T_2 be an A-computable 2-branching subtree of T_1. Then $T_3 = \{\tau \in T_0 : \Psi(\tau) \in T_2\}$ is an A-computable 2-branching subtree of T_0.*

Proof. The proof is not difficult and is left to the reader. $\square$

So now let us suppose that we wish to construct a strong minimal cover for $deg(A)$. In order to do so we must construct $B \geq_T A$ and satisfy all requirements;

$\mathcal{R}_i : \Psi_i(B)$ total $\rightarrow (\Psi_i(B) \leq_T A$ or $B \leq_T \Psi_i(B))$
$\mathcal{P}_i : B \neq \Psi_i(A)$

In order to ensure that $B \geq_T A$ we can simply insist that B should be an $A\oplus$-compatible string. Thus we begin with the restriction that B should lie on the tree T_0 containing all strings of even length which are $A\oplus$-compatible, an A-computable 2-branching tree.

In order to meet all other requirements we might try to proceed by finite extension. We define B_0 to be the empty string. Suppose that by the end of stage s we have defined $B_s \in 2^{<\omega}$ on T_s, an A-computable 2-branching tree, in such a way that if B extends B_s and lies on T_s then all requirements

$\mathcal{R}_i, \mathcal{P}_i$ for $i < s$ will be satisfied. At stage $s + 1$ we might proceed, initially, just as if we were only trying to construct a minimal cover for A. We can assume that $\Psi_s = \hat{\Psi}_s$ (otherwise replace Ψ_s with $\hat{\Psi}_s$ in what follows). We ask the question, "does there exist $\tau \supseteq B_s$ on T_s such that no two strings on T_s extending τ are a Ψ_s-splitting?".

If so: then let τ be such a string. We can define $T_{s+1} = T_s$ and (just to make the satisfaction of $\mathcal{P}_s$ explicit) define B_{s+1} to be some extension of τ on T_s sufficient to ensure $\mathcal{P}_s$ is satisfied.

If not: then we can define T'_s to be an A-computable 2-branching Ψ_s-splitting subtree of T_s having B_s as least element—the idea being that we shall eventually define T_{s+1} to be some subtree of T'_s. If B lies on T'_s then we shall have that $B \leq_T \Psi_s(B) \oplus A$. Of course this does not suffice, since for the satisfaction of $\mathcal{R}_s$ we require that $B \leq_T \Psi_s(B)$. Suppose, however, that we know A satisfies the property;

($\dagger$) If T is an A-computable 2-branching tree, then there exists T' a subtree of T which is also an A-computable 2-branching tree and which satisfies the property that if $C \in [T']$ then $A \leq_T C$.

Lemma 5.1 then suffices to ensure that we can define T_{s+1} to be a subtree of T'_s which is an A-computable 2-branching tree, and which satisfies the property that if $B \in [T_{s+1}]$ then $A \leq_T \Psi_s(B)$ so that, since $B \leq_T \Psi_s(B) \oplus A$, $B \leq_T \Psi_s(B)$. Then we can define B_{s+1} to be an extension of B_s lying on T_{s+1} sufficient to ensure the satisfaction of $\mathcal{P}_s$.

In conclusion, then, if A satisfies the property ($\dagger$) we can construct a strong minimal cover for $deg(A)$.

6. The proof of theorem 4.3

The remarks of the last section suffice to show that in order to prove theorem 4.3 we need only prove that if A is of **0**-dominated degree which is not FPF then A satisfies ($\dagger$). So let us now assume that A is such a set and that we are given a Turing functional Φ which satisfies the property that for all σ, $\Phi(A; \sigma) \downarrow \in \{0, 1\}$ and $\Phi(A; \sigma) = 1$ iff $\sigma \in T$, where T is A-computable and 2-branching. We can assume that T has a single element of level 0 which is the empty string and that $\Phi(\tau; \sigma) \downarrow$ only if the computation converges in $\leq |\tau|$ steps and $\Phi(\tau; \sigma') \downarrow$ for all σ' such that $|\sigma'| < |\sigma|$.

Definition 6.1. *For all τ we define $T(\tau) = \{\sigma : \Phi(\tau; \sigma) \downarrow = 1\}$.*

It will be convenient, also, to assume that for any τ and $\sigma \in T(\tau)$, σ has at most two successors in $T(\tau)$ and that any string of level 0 in $T(\tau)$ must be λ.

Consider now the A-computable function g defined as follows; for every n, $g(n)$ is the greatest value $|\sigma|$ such that σ is of level n in T. Since A is of **0**-dominated degree we can take computable and increasing f which majorizes g.

Definition 6.2. *We denote $\Omega(\tau, n)$ iff $n = 0$ or:*
- *$T(\tau)$ is of level at least n and is 2-branching below level n, and*
- *for every $n' \leq n$, the greatest value $|\sigma|$ such that σ is of level n' in $T(\tau)$ is $\leq f(n')$.*

6.1. **Defining Π.** We define Π by enumeration in stages. Initially Π contains only the empty string. At stage $s > 0$ we consider all strings τ of length s and for each such string we proceed as follows. Let n be the greatest such that $\Omega(\tau', n)$ for $\tau' \subset \tau$ such that $\tau' \in \Pi$. If it is the case that $\Omega(\tau, n')$ for some $n' \geq n + 2$ and all strings in $T(\tau)$ of level n' are of length at least $f(n + 2)$ then enumerate τ into Π.

6.2. **Using Π-thin Π'.** Let Π' be Π-thin and let $\Lambda = \{\tau_i : 1 \leq i \leq k\}$ be a prefix-free set of strings in Π' such that each τ_i extends $\tau \in \Pi'$. For each i let n_{τ_i} be the greatest n such that $\Omega(\tau_i, n)$ and let n_τ be the greatest n such that $\Omega(\tau, n)$. We show that if σ is any string in $T(\tau)$ of level n_τ then we can choose two strings extending σ of level n_{τ_i} from each $T(\tau_i)$, $\sigma_{i,0}$ and $\sigma_{i,1}$ say, so that if $i \neq i'$ or $j \neq j'$ then $\sigma_{i,j} | \sigma_{i',j'}$.

Define $m_0 = |\tau|_\Pi$. For each $m \geq 1$ let Λ_m be the set of τ_i which are of level $\leq m_0 + m$ in Π and let $\Lambda_m^\star$ be the set of τ_i which are of level $m_0 + m$ in Π. Defining $r_m = \Sigma_{m'=1}^m 2^{-m'} |\Lambda_{m'}^\star|$ we have that $r_m \leq 1$. We show by induction on m that we can choose two strings $\sigma_{i,0}$ and $\sigma_{i,1}$ extending σ of level n_{τ_i} from each $T(\tau_i)$ such that $\tau_i \in \Lambda_m$ and $(1 - r_m)2^{m+1}$ different strings $\psi_{i,j}$ extending σ of level $n_\tau + 2m$ from each $T(\tau_i)$ for $\tau_i \in \Lambda - \Lambda_m$ in such a way that (where these values are defined):

- if $i \neq i'$ or $j \neq j'$ then $\sigma_{i,j} | \sigma_{i',j'}$, and
- for any i, j, i', j' we have $\sigma_{i,j} | \psi_{i',j'}$.

Case $m = 1$. If $|\Lambda_1^\star| = 0$ then the result is clear, so suppose first that $|\Lambda_1^\star| = 1$ and (simply for the sake of simplicity of labeling) let us suppose that $\tau_1 \in \Lambda_1^\star$. We choose any two different strings from $T(\tau_1)$ of level n_{τ_1} extending σ and define these to be $\sigma_{1,0}$ and $\sigma_{1,1}$. Observe that every string in $T(\tau_1)$ of level n_{τ_1} is of length $\geq f(n_\tau + 2)$. Now consider those $\tau_i \in \Lambda - \Lambda_1^\star$. Since every string in $T(\tau_i)$ of level $n_\tau + 2$ is of length $\leq f(n_\tau + 2)$ there are at most two strings in $T(\tau_i)$ of level $n_\tau + 2$ which are compatible with either $\sigma_{1,0}$ or $\sigma_{1,1}$. We can therefore define $\psi_{i,0}$ and $\psi_{i,1}$ as required.

Suppose $|\Lambda_1^\star| = 2$ and $\tau_1, \tau_2 \in \Lambda_1^\star$. First we choose any two different strings from $T(\tau_1)$ of level n_{τ_1} extending σ and define these to be $\sigma_{1,0}$ and $\sigma_{1,1}$. Since every string in $T(\tau_2)$ of level $n_\tau + 2$ is of length $\leq f(n_\tau + 2)$ there are at most two strings in $T(\tau_2)$ of level $n_\tau + 2$ which are compatible with either $\sigma_{1,0}$ or $\sigma_{1,1}$. We can therefore define $\sigma_{2,0}$ and $\sigma_{2,1}$ as required.
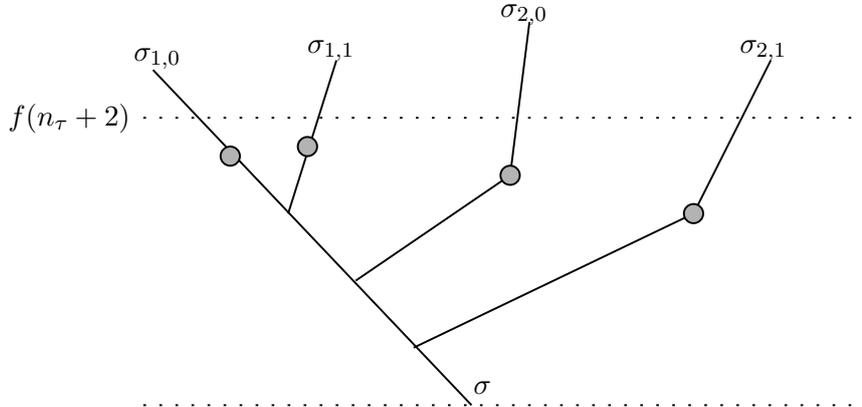

Figure 1

The diagram illustrates what happens in the case that $|\Lambda_1^\star| = 2$. First we pick $\sigma_{1,0}$ and $\sigma_{1,1}$. These are strings from $T(\tau_1)$ of level n_{τ_1} extending σ, and are therefore of length $\geq f(n_\tau + 2)$. The coloured circles indicate what the strings extending σ and of level $n_\tau + 2$ may look like in $T(\tau_2)$. These strings are of length $\leq f(n_\tau + 2)$ and therefore at most two of them are compatible with either of the strings $\sigma_{1,0}$, $\sigma_{1,1}$. We may therefore choose $\sigma_{2,0}$ and $\sigma_{2,1}$ of level n_{τ_2} in $T(\tau_2)$, which are incompatible with $\sigma_{1,0}$ and $\sigma_{1,1}$. These strings will be of length $\geq f(n_\tau + 2)$.

Case $m > 1$. By the induction hypothesis we can choose two strings $\sigma_{i,0}$ and $\sigma_{i,1}$ extending σ of level n_{τ_i} from each $T(\tau_i)$ such that $\tau_i \in \Lambda_{m-1}$ and $(1 - r_{m-1})2^m$ different strings $\psi_{i,j}$ extending σ of level $n_\tau + 2(m-1)$ from each $T(\tau_i)$ for $\tau_i \in \Lambda - \Lambda_{m-1}$ in such a way that if $i \neq i'$ or $j \neq j'$ then $\sigma_{i,j} | \sigma_{i',j'}$ and for any i, j, i', j' we have $\sigma_{i,j} | \psi_{i',j'}$. For each $\tau_i \in \Lambda - \Lambda_{m-1}$ take the four extensions of each $\psi_{i,j}$ of level $n_\tau + 2m$ in $T(\tau_i)$ and relabel so that these are the strings $\psi_{i,j}$. There are at most $(1 - r_{m-1})2^m$ strings in $\Lambda_m^\star$. We proceed first by defining in turn the strings $\sigma_{i,j}$ such that $\tau_i \in \Lambda_m^\star$, from amongst the extensions of the strings $\psi_{i,j'}$. Whenever we define such $\sigma_{i,j}$ it is of length $\geq f(n_\tau + 2m)$ and it is therefore the case that for each $\tau_{i'} \in \Lambda - \Lambda_{m-1}$ there is at most one string $\psi_{i',j'}$ which is compatible with $\sigma_{i,j}$. Since we must choose at most $(1 - r_{m-1})2^{m+1}$ strings $\sigma_{i,j}$ and each $\tau_{i'} \in \Lambda - \Lambda_{m-1}$ has $(1 - r_{m-1})2^{m+2}$ incompatible $\psi_{i',j'}$ we can define all the $\sigma_{i,j}$ as required. This leaves each $\tau_i \in \Lambda - \Lambda_m$ with at least $(1 - r_{m-1})2^{m+1} \geq (1 - r_m)2^{m+1}$ strings $\psi_{i,j}$ which are incompatible with any $\sigma_{i',j'}$.

6.3. **Defining** T'. Let c.e. Π' be Π-thin with $A \in [\Pi']$. Since A is of **0**-dominated degree we can let $\Pi^\star$ be a subset of Π' such that:

- $A \in [\Pi^\star]$, and $\Pi^\star$ has as element of level 0 the empty string λ,
- each $\tau \in \Pi^\star$ has a finite number of successors, and
- there is a computable function which given any τ such that $\tau \in \Pi^\star$ returns m such that D_m (the mth finite set according to some fixed effective listing) codes the successors of τ in $\Pi^\star$.

Given a computable enumeration $\{\Pi_s^\star\}_{s \geq 0}$ satisfying

(1) $\Pi_0^\star = \{\lambda\}$,
(2) if $\tau \in \Pi_{s+1}^\star - \Pi_s^\star$ then τ extends a leaf of $\Pi_s^\star$, and
(3) if $\tau, \tau' \in \Pi_{s+1}^\star - \Pi_s^\star$ then these strings are incompatible,

we proceed in an effective fashion to enumerate values $T'(\tau)$ and axioms for Θ such that $T' = \bigcup \{T'(\tau) : T'(\tau) \downarrow, \tau \subset A\}$ is an A-computable 2-branching subtree of T, and for all $C \in [T']$ we have $\Theta(C) = A$. This suffices, then, to show that A satisfies (†), as required.

Stage 0. We define $T'(\lambda) = \{\lambda\}$.

Stage s+1. We can assume that strings are enumerated into $\Pi_{s+1}^\star$ extending precisely one leaf of $\Pi_s^\star$, τ say, which is a string of level m (say) in $\Pi^\star$. Let the strings enumerated into $\Pi_{s+1}^\star$ extending τ be $\tau_1, .., \tau_k$. For each i let n_{τ_i} be the greatest n such that $\Omega(\tau_i, n)$ and let n_τ be the greatest n such that $\Omega(\tau, n)$. We will have already defined the value $T'(\tau)$ to be a tree of level m, which is 2-branching below level m, and with each leaf a string of level n_τ in $T(\tau)$.

Now we must define each $T'(\tau_i)$ to be a tree of level $m + 1$ which is 2-branching below level $m + 1$, with $T'(\tau)$ as subtree and with two leaves extending each leaf σ of $T'(\tau)$, with each leaf a string of level n_{τ_i} in $T(\tau_i)$, and such that for $i \neq i'$ any leaf of $T'(\tau_i)$ is incompatible with any leaf of $T'(\tau_{i'})$. Thus for each leaf σ of $T'(\tau)$ we must choose for each τ_i two extensions $\sigma_{i,0}$ and $\sigma_{i,1}$ of level n_{τ_i} in $T(\tau_i)$ in such a way that $\sigma_{i,j} | \sigma_{i',j'}$ if either $i \neq i'$ or $j \neq j'$. The observation of section 6.2 says precisely that this is possible. Since these strings are pairwise incompatible we can then consistently define $\Theta(\sigma') = \tau_i$, for each σ' which we have just defined as a leaf of $T'(\tau_i)$.

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